Incomplete markets

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Preface

Financial risks can be managed by trading in financial markets where investors can exchange uncertain future cash-flows. By appropriate trading, individuals and financial institutions may modify their net cash-flows to better conform to their risk preferences. For example, a home owner may be able to achieve a more attractive risk profile for his future cash-flows by buying a home insurance. Insurers, on the other hand, invest insurance premiums in financial markets in order to optimize their net cash-flow structure resulting from paying insurance claims and collecting investment returns. The same principle is behind the classical Black–Scholes–Merton option pricing framework, where the seller of an option invests the premium in financial markets according to an investment strategy whose return matches the option payout.

In complete markets, prices of contingent claims are uniquely determined by replication arguments but in incomplete markets, subjective factors such as market expectations and risk preferences come in. This is pronounced in insurance contracts, credit derivatives, weather derivatives, energy contracts, etc. whose cash-flows are often largely unrelated to liquidly traded securities. Moreover, in incomplete markets, there are two distinct notions often referred to as the "price":

1. The least amount of cash that would allow for hedging a claim at an acceptable level of risk.

2. The least amount of cash one could sell a claim for without worsening one's risk-return profile.

The first notion is important in *accounting* while the second one is more relevant in *trading* and actual business transactions. The first notion corresponds to the option pricing formula of Black and Scholes [4] who studied claims whose payoffs can be replicated exactly by an appropriate investment strategy. It is also used in various accounting standards and supervisory frameworks for defining "book values" of financial liabilities. In banking, it is sometimes called the "economic capital" while in insurance, the terms "technical provisions" and "reserving" are often used; see e.g. Article 76 of the Solvency II Directive 2009/138/EC of the European Parliament. Such a book value is not necessarily a price at which an agent would be willing to sell or buy a claim.

Offered prices are better described by the second notion which describes the *indifference principle*. For example, a seller of a financial product charges (at least) a price that allows him to sell the product without worsening the risk-return profile of his existing financial position. Indifference pricing has been widely applied in various areas of finance and insurance; see e.g. Bühlmann [5], Hodges and Neuberger [21] or Bielecki et al. [3] for a small sample. We refer the reader to Carmona [6] for further references on the topic.

These notes give an introduction to asset-liability management, accounting and indifference pricing in terms of basic optimization theory. Our aim is to give a unified treatment of the above concepts and to study their relations and basic properties with minimal mathematical sophistication. Mathematical techniques are introduced only when they become necessary for the development of the theory. Working with discrete-time models allows us to avoid many of the technicalities associated with continuous time models. This leaves room for practical considerations that are sometimes neglected in more mathematical texts. In particular, we extend the classical theory of mathematical finance by allowing for nonlinear illiquidity effects and portfolio constraints. Such features are significant in practice but they invalidate much of the classical theory. We deviate from the classical theory also in that we do not insist on the existence of a perfectly liquid numeraire asset. This means that one can no longer postpone payments by shorting the numeraire so the payment schedule of a financial contract becomes an important issue. This is essential in practice where much of trading consists of exchanging sequences of cash-flows. Examples include loans as well as various swap and insurance contracts where both claims and premiums involve payments at several points in time.

Traditionally, the main tools in mathematical finance have come from stochastic analysis but in incomplete markets, techniques of convex analysis are required. Convexity is often indispensable in mathematical and numerical analysis of financial optimization problems. For example, general characterizations of the *no-arbitrage* property of a perfectly liquid market model in terms of *martingale measures* is largely based on separation theorems for convex sets; see Föllmer and Schied [17] and Delbaen and Schachermayer [12] for comprehensive study of the classical linear model of financial markets. Techniques of convex analysis allow also for extending the classical theory of mathematical finance to more realistic models with portfolio constraints and illiquidity effects; see for example Dermody and Rockafellar [13, 14], Cvitanic and Karatzas [8, 9], Jouini and Kallal [24, 23], Kabanov [25], Föllmer and Schied [17, Chapter 9], Evstigneev, Schürger and Taksar [16], Schachermayer [46], Çetin and Rogers [7], Kabanov and Safarian [26] and Pennanen [35, 37]. Moreover, the optimization perspective brings in variational and computational techniques that have been successful in other fields of applied mathematics such as partial differential equations and operations research.

The first part of the notes introduces the main concepts in a simple one-period model. The second part extends the theory to a multiperiod setting with nonlinear illiquidity effects. Basic principles in contingent claim valuation are derived by purely algebraic arguments using elementary convex analysis.

The third part develops the duality theory that is behind e.g. martingale representations of contingent claim prices in the classical linear model.

Chapter 1

Single-period models

This chapter studies the basics of financial risk management in a simple single-period model of perfectly liquid markets. We consider a finite set of traded assets and assume (like in the Black–Scholes model) that arbitrary amounts of all assets can be bought or sold at given unit prices. While today's prices are assumed known, tomorrow's prices will be modelled as random variables in a general probability space. Already in this simple setting, one can develop the main principles of financial risk management and contingent claim valuation in incomplete financial markets.

The main characteristic (and difficulty) of *incomplete financial markets* is that, unlike in complete market models such as the famous Black–Scholes model, not all contingent claims can be perfectly replicated by trading in the market. As a result, the seller of a contingent claim will usually be exposed to the risk that the price charged for the claim may not be enough to cover the claim in all future scenarios. In other scenarios, on the other hand, there may be surplus. Such a situation is the rule rather than an exception in practice. Whether a random net position is acceptable to an agent depends on the agent's risk preferences. It follows that in incomplete markets (in practice) *values of financial contracts are subjective*. Indeed, the values depend on an agent's risk preferences, financial position as well as his views concerning the future development of financial markets. This subjectivity is the driving force in financial markets since different valuations give incentives for financial agents to trade with each other. The classical complete markets theory of financial mathematics is unable to explain the subjectivity and thus the existence of financial markets in the first place.

Another important feature of incomplete markets is that the notion of a "price" of a contingent claim has (at least) two meaningful generalizations. The first one has an important role in accounting and financial supervision while the second is more relevant in trading. It turns out that in complete market models the two notions coincide, which explains why the distinction is rarely made in the traditional financial mathematics literature.

These notes develop a unified approach where contingent claim valuation and risk management are based on a single asset-liability management (ALM) model. The ALM-model is an optimization problem of finding an investment strategy that optimizes a given risk measure of the unhedged part of an agent's financial liabilities. After a brief introduction to numerical representation of risk preferences, the basic single-period ALM-model is presented in Section 1.3 and the two notions of contingent claim valuation are studied in Sections 1.4 and 1.5, respectively. Section 1.6 takes a closer look at modern securities exchanges and concludes that real trading costs are nonlinear unlike in the classical perfectly liquid market models studied in Sections 1.3-1.5 and elsewhere in the literature. Chapter 2 extends the analysis to more realistic multiperiod market models with illiquidity effects.

1.1 Financial mathematics

In financial mathematics, it is common to model uncertain quantities such as future investment returns and payouts of financial products as *random variables*. The probabilistic description should describe an investor's *views* concerning the uncertain future. Building such a model for uncertain financial quantities often builds on statistical analysis of historical data, current market conditions as well as on expert knowledge on the functioning of financial markets and the economy more generally.

Given a probabilistic description of investment returns on the relevant asset classes and of other cash-flows, one can analyse the distributional properties of the profit and loss (P&L) (the future net wealth of an investor) associated with a given *investment strategy*. A natural problem then is to find an investment strategy that yields a P&L as "nice" as possible. To formulate the problem mathematically, one needs a measure of "niceness" of a random P&L. Such a description depends on an agent's risk preferences which is an elusive and highly *subjective* notion. Nevertheless, the subjective risk preferences drive the trading decisions of every agent. A quantitative representation of risk preferences is a real-valued function on the space of random variables (the probabilistic descriptions of P&Ls). This will be the topic of the next section.

Given a probabilistic description of the relevant risk factors and a quantitative representation of risk preferences, one arrives at a mathematical *optimization problem* of finding an investment strategy that yields the best possible value of the function representing risk preferences. Such problems are, in general, difficult to solve and often require *numerical techniques*. Nevertheless, it is possible deduce many important properties about the problem through mathematical analysis without solving them explicitly. This is the goal of these notes.

Both numerical solution and mathematical analysis of optimization problems are greatly facilitated when the problem is *convex*. For convex optimization problems efficient numerical algorithms and software are readily available. The analysis of such problems falls into the field of *convex analysis*. These notes will develop the necessary parts of the theory as needed.

1.2 Risk preferences

We start our study of incomplete financial markets by a brief review of mathematical description of preferences concerning uncertain future cash-flows. The preferences will be an important part of the ALM models studied in the next sections. We will not aim at full generality nor at comprehensive study of the notions introduced; references to more complete texts on the topic are given at the end of this section. The notions introduced here suffice for the financial analysis in later sections.

Consider an economic agent who at present time $t = 0$ is faced with the liability of paying c units of cash at some future time $t = 1$. We will study situations where the amount c may be uncertain until the time it is paid. We will be interested in quantifying the agents preferences over uncertain payments. As usual, we will model uncertain quantities as *random variables* (that is, real-valued measurable functions) on a probability space $(\Omega, \mathcal{F}, P)$. We allow c to take arbitrary real values so it may describe expenses as well as income. Indeed, negative values of c are interpreted as income.

We will not distinguish between random variables that take the same values with probability one. More precisely, random variables c and $\tilde{c}$ are said to be *equivalent* if they are equal with probability one; i.e. if $P(\{\omega \in \Omega \,|\, c(\omega) = \tilde{c}(\omega)\}) = 1$. The linear space of equivalence classes of real-valued random variables will be denoted by $L^0 := L^0(\Omega, \mathcal{F}, P)$. In particular, we model liabilities c as elements of the space L^0.

We will describe the agent's preferences concerning random liabilities by a function $\mathcal{V} : L^0 \to \overline{\mathbb{R}}$: the agent prefers a liability $c^1 \in L^0$ over another $c^2 \in L^0$ if $\mathcal{V}(c^1) < \mathcal{V}(c^2)$, while the agent is indifferent between c^1 and c^2 if $\mathcal{V}(c^1) = \mathcal{V}(c^2)$. Throughout these notes, $\overline{\mathbb{R}}$ denotes the extended real line $\mathbb{R} \cup \{+\infty, -\infty\}$. The function $\mathcal{V}$ is called a *numerical representation of risk preferences*. It can be thought of as a numerical measure of "unhappiness" an agent experiences at time $t = 0$ when facing a random payment c at time $t = 1$. Note that the preference relations are not affected if we multiply $\mathcal{V}$ by a positive constant or add a constant to it: given any constants $a > 0$ and b, we have $\mathcal{V}(c^1) < \mathcal{V}(c^2)$ if and only if $a\mathcal{V}(c^1) + b < a\mathcal{V}(c^2) + b$.

Various choices for $\mathcal{V}$ can be considered. Two extremes would be the *expectation* $\mathcal{V}(c) = Ec$ and the *essential supremum*[1] $\mathcal{V}(c) = \operatorname{ess\,sup} c$. The former corresponds to a "risk neutral" agent who is only concerned about the average payment, not about its uncertainty. The essential supremum corresponds to an agent who only looks at the worst case.

It is natural to assume that $\mathcal{V}$ is *nondecreasing* in the sense that $\mathcal{V}(c^1) \le \mathcal{V}(c^2)$ if $c^1 \le c^2$ almost surely. Indeed, violation of this condition would mean that the agent might be happy to throw away cash. The agent is said to be *risk averse* if $\mathcal{V}(Ec) \le \mathcal{V}(c)$ for all $c \in L^1$. In other words, a risk averse agent prefers known payments to uncertain ones if they have the same mean.

[1] Recall that $\operatorname{ess\,sup} c := \inf\{\alpha \in \mathbb{R} \,|\, c \le \alpha \ P\text{-a.s.}\}$.

Example 1.1 (Expected utility) *Let v be a nondecreasing function on $\mathbb{R}$ and define*

$$\mathcal{V}(c) := E[v(c)],$$

where E denotes the expectation[2]*. The idea is that $v(c)$ quantifies the unhappiness of the agent when making a payment at time $t = 1$. Considering random payments at time $t = 0$ before knowing the value of c to be paid at time $t = 1$, the agent simply takes the expectation of his unhappiness $v(c)$.*

It is natural to assume that the marginal disutility $v'(c)$ increases with the claim size c. Indeed, you may be able to cover low values of c by reducing your spending on nonessential things but when c gets larger, you may need to give up something essential. If v' is nondecreasing, then the function v is a convex *function on $\mathbb{R}$; see Exercise 1. This in turn implies that $\mathcal{V}$ is a convex function on L^0; see Exercise 2. By Jensen's inequality, convexity of v also implies*

$$v(Ec) \leq Ev(c)$$

for all $c \in L^1$, so $\mathcal{V}$ is risk averse. When the function v is twice differentiable, the number

$$\rho(\alpha) = \frac{v''(\alpha)}{v'(\alpha)}$$

is known as the Arrow–Pratt coefficient of absolute risk aversion of v at level $\alpha \in \mathbb{R}$. It measures the "curvature" of the function v. Note that ρ is not affected if we multiply v by a positive constant or add a constant to it. Noting that $\rho(\alpha) = (\ln v')'(\alpha)$, we see, in particular, that if $\rho(\alpha)$ is a positive constant function, then $v(c) = ae^{\rho c} + b$ for constants $a > 0$ and b. (How would you estimate your utility function?)

Representation of risk preferences by an expectation of the form $Ev(c)$ was proposed by Daniel Bernoulli already in 1738 in the context of gambling; see Exercise 3. Sufficient conditions for the existence of such a representation of risk preferences were given by von Neumann and Morgenstern [48], hence the name von Neumann and Morgenstern representation*. In this context, the function $x \mapsto -v(-x)$ is known as the* utility function*.*

Example 1.2 (Mean-variance) *The famous* Capital Asset Pricing Model *is based on the mean-variance criterion given by $\mathcal{V}(c) = Ec + \lambda\sigma^2(c)$ where λ is a positive scalar and $\sigma^2(c)$ is the variance of c; see Exercise 6 in Section 1.3. The mean-variance criterion was studied already in the PhD-thesis of Harry Markowitz [32], who together with Merton Miller and William Sharpe received the Nobel price in economics in 1990. The mean-variance criterion is, however, irrational since it is not nondecreasing. Indeed, the variance penalizes both the upside as well as the downside.*

[2]Throughout these notes, we define the expectation of an extended real-valued random variable c as $+\infty$ unless the positive part of c has finite expectation. The expectation is then well-defined for any random variable. This convention of defining extended real-valued integrals is not arbitrary but specifically suited for studying minimization problems.

Example 1.3 (Value at Risk) *In the financial context, the* lower δ-quantile

$$\mathcal{V}(c) = \inf\{\alpha \in \mathbb{R} \,|\, P(c \leq \alpha) \geq \delta\}$$

is known as Value at Risk, *V@R for short, at level $\delta \in (0,1)$. It gives the least amount of cash α one needs to subtract from the random claim c so that $c - \alpha$ is nonpositive with probability δ. V@R is in widespread use in financial risk management and supervision. However, it has been criticised for its lack of risk aversion and convexity. Discussions on replacing it with more "coherent" measures of risk are underway.*

Example 1.4 (Risk measures) *This example looks at a whole class of numerical representations of risk preferences. It goes under the name of "convex risk measure" which is a term used to refer to any function satisfying a set of three axioms. The notion has gained wide popularity in literature and practice but as we will see, only the first two axioms are needed for the general theory.*

A function $\mathcal{R} : L^\infty \to \mathbb{R}$ is said to be a convex risk measure[3] *if*

1. *$\mathcal{R}$ is convex*
2. *$\mathcal{R}$ is nondecreasing: $\mathcal{R}(c_1) \leq \mathcal{R}(c_2)$ whenever $c_1 \leq c_2$ almost surely*
3. *$\mathcal{R}(c + \alpha) = \mathcal{R}(c) + \alpha$ for every constant $\alpha \in \mathbb{R}$.*

Receiving $\mathcal{R}(c)$ units of cash at time $t = 1$ compensates for delivering c in the sense that the risk of the net cash-flow $c - \mathcal{R}(c)$ is zero (by the third property). We will see a more practical interpretation of a risk measure in the next section.

Given a convex risk measure, one can extend it to a nondecreasing convex function on all of L^0 by defining

$$\mathcal{V}(c) = \inf_{d \in L^\infty} \{\mathcal{R}(d) \,|\, c \leq d\};$$

see Exercise 8 in Section 1.4.

Exercises

A set X is said to be a *linear space* (or a vector space) if $x_1 + x_2 \in X$ and $\alpha x_1 \in X$ whenever $x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$ and if the addition and multiplication operations satisfy the usual associativity and commutativity conditions. The meaning of the addition and scalar multiplication depend on the context. For example the space $\mathbb{R}^J$ is a linear space when addition and scalar multiplication are defined componentwise. The space $L^0(\Omega, \mathcal{F}, P)$ of random variables is a vector space if addition and scalar multiplication are defined scenariowise (ω by ω).

A set C in a linear space X is *convex* if

$$\alpha_1 x_1 + \alpha_2 x_2 \in C$$

[3] Recall that $L^\infty := \{c \in L^0 \mid \operatorname{ess\,sup} |c| < \infty\}$.

whenever $x_1, x_2 \in C$ and $\alpha_1, \alpha_2 > 0$ are such that $\alpha_1 + \alpha_2 = 1$. An extended real-valued function f on X is *convex* if its *epigraph* $\operatorname{epi} f := \{(x,\alpha) \,|\, f(x) \leq \alpha\}$ is a convex set in $X \times \mathbb{R}$. It is not hard to check that a function f is convex iff

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

whenever $x_1, x_2 \in \operatorname{dom} f$ and $\alpha_1, \alpha_2 > 0$ are such that $\alpha_1 + \alpha_2 = 1$. Here $\operatorname{dom} f := \{x \in X \,|\, f(x) < \infty\}$ is the *effective domain* of f. A function f is *concave* if $-f$ is convex. The *lower level sets* $\operatorname{lev}_\alpha f := \{x \,|\, f(x) \leq \alpha\}$ of a convex function are convex sets but the converse does not hold in general. An extended real-valued function is said to be *proper* if it never takes the value $-\infty$ and it is not identically equal to $+\infty$.

1. Let $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ be nondecreasing and $\bar{x} \in \mathbb{R}$ such that $\phi(\bar{x})$ is finite. Show that the function

$$f(x) = \int_{\bar{x}}^{x} \phi(z) dz$$

   is convex.

2. Let $v : \mathbb{R} \times \Omega \to \overline{\mathbb{R}}$ be an $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable function such that $v(\cdot, \omega)$ is nondecreasing and convex for every $\omega \in \Omega$. Show that

$$\mathcal{V}(c) = Ev(c) := \int_\Omega v(c(\omega), \omega) dP(\omega)$$

   is a nondecreasing convex function on L^0. Again, the integral is defined as $+\infty$ unless the positive part of the integrand is integrable.

3. In the context of Example 1.1, the *certainty equivalent* of a claim $c \in L^0$ is defined as the greatest number $CE(c)$ such that $v(CE(c)) = Ev(c)$. The interpretation is that the agent would be indifferent between paying the random claim c or the fixed amount $CE(c)$. If v is strictly increasing, then v is invertible and $CE(c) = v^{-1}(Ev(c))$.

   (a) Show that if v is nondecreasing and convex, then $CE(c) \geq Ec$. The difference $CE(c) - Ec$ is sometimes called the "risk premium".

   (b) Consider a gamble where a coin is tossed until the first heads appears. If this happens at the nth toss, you win 2^{n-1} pounds (that is, the claim c is -2^{n-1}). Calculate the certainty equivalent when

      i. $v(c) = c$,

      ii. $v(c) = -\ln(-c)$.

      The latter certainty equivalent was proposed by Daniel Bernoulli in 1738 to "value" the gamble.

Further reading

Numerical representations $\mathcal{V}$ of risk preferences can be seen as extensions of risk measures (see Example 1.4) introduced in Artzner, Delbaen, Eber and Heath [1]; see e.g. Föllmer and Schied [17] or Rockafellar [43] for general discussions on risk measures and description of risk preferences. Unlike a risk measure, as defined in the above references, the values of $\mathcal{V}$ need not represent monetary values of uncertain cash-flows. Such values will be studied in Sections 1.4 and 1.5 below. We refer the reader to Kreps [30] and Föllmer and Schied [17, Chapter 2] for general study of risk preferences and their numerical representations.

Classical references on the analysis of convex functions and sets include the lecture notes of Moreau [34] and the book of Rockafellar [40]. Although [40] is concerned with finite-dimensional spaces, many of the facts found there extend to more general vector spaces; see e.g. [41, 15].

1.3 Asset-liability management

Consider a financial market with a finite set J of assets that can be traded at two dates, the present time $t = 0$ and time $t = 1$ in the future. For simplicity, we assume in Sections 1.3–1.5 that the market is perfectly liquid so that unit prices of the assets do not depend on our trades. The unit price of asset $j \in J$ at time t will be denoted by s_t^j. We assume that the vector $s_0 = (s_0^j)_{j\in J}$ of current prices is known to us before we trade at time $t = 0$. The price vector $s_1 = (s_1^j)_{j\in J}$ will remain uncertain until we trade at time $t = 1$. We model s_1 as a *random* vector on a probability space $(\Omega, \mathcal{F}, P)$. That is, $s_1^j \in L^0$ for each $j \in J$.

Buying a portfolio $x = (x^j)_{j\in J}$ of assets at time $t = 0$ costs

$$s_0 \cdot x := \sum_{j\in J} s_0^j x^j$$

units of cash. Here x^j denotes the number of units of asset $j \in J$ bought. If we hold on to the portfolio, it will be worth $s_1 \cdot x$ at time $t = 1$. Clearly, $s_1 \cdot x$ is random since s_1 is random.

Remark 1.5 *The portfolio held over* $[0, T]$ *can also be described in terms of units of cash invested. Indeed, buying buying* x^j *units of asset* j *means that we invest* $h^j = s_0^j x^j$ *units of cash in it. The value of the portfolio at time* $t = 1$ *can then be expressed as* $r \cdot h$*, where* $r = (r^j)_{j\in J}$ *is the random vector with the* returns $r^j = s_1^j / s_0^j$ *as its components; see Exercises 4 and 6 below. While in practice, it is more common to describe investments in terms of* h*, formulations in terms of* x *are often more convenient in mathematical analysis and thus more common in financial mathematics.*

Consider an economic agent whose financial position is described by an initial wealth of w units of cash at time $t = 0$ and a financial liability of delivering

$c \in L^0$ (a *claim*) units of cash at time $t = 1$. We allow c to take arbitrary real values so it may describe expenses as well as income. The agent may invest his wealth w in financial markets and use the proceeds to cover part of the liability. Buying a portfolio $x \in \mathbb{R}^J$ at time $t = 0$, the agent will then just need to come up with the residual $c - s_1 \cdot x$ payment. The agent's investment problem can be written as the *optimization problem*

$$\begin{array}{lll} \text{minimize} & \mathcal{V}(c - s_1 \cdot x) \quad \text{over} & x \in D, \\ \text{subject to} & s_0 \cdot x \leq w, & \end{array} \tag{ALM}$$

where $D \subseteq \mathbb{R}^J$ describes *portfolio constraints* and the function $\mathcal{V} : L^0 \to \overline{\mathbb{R}}$ is a numerical representation of the agent's risk preferences (see the previous section) concerning the net expenditure $c - s_1 \cdot x$ at time $t = 1$. The constraint $x \in D$ can be used to describe various restrictions posed on the portfolios the agent is able/allowed to hold. In practice, all assets are subject to some restrictions. Short selling constraints correspond to $D = \mathbb{R}^J_+$, while in unconstrained models, $\mathcal{D} = \mathbb{R}^J$; see Exercise 7 for more examples.

To emphasize the role of the liability c, we will call (ALM) the *asset-liability management* problem. Asset-liability management refers to the general problem of managing assets so that their proceeds cover given liabilities as well as possible. Such problems are fundamental in finance, from risk management to contingent claim valuation. Traditionally (both in theory and practice), pricing and *hedging* of liabilities has been seen as a separate problem from that of optimal investment, but as we will see in later sections, (ALM) unifies the two theories under a single coherent framework. Of course, asset-liability management problems in practice are usually more complicated than (ALM). Problem (ALM) is the simplest mathematical model that allows us to develop the basic principles of financial risk management and contingent claim valuation in incomplete markets. More realistic models are treated in Chapter 2.

Problem (ALM) depends on an agent's

1. views described by the probability measure P,

2. risk preferences described the disutility function $\mathcal{V}$,

3. financial position described by the initial wealth and liabilities (w, c).

All these factors are *subjective*. Agents with different views, preferences or financial positions may have very different investment strategies and, as we will see, different valuations of financial contracts. This is a significant departure from the asset pricing theory in complete markets (such as the Black–Scholes–Merton model) where prices are not affected by subjective factors. We will see, however, that such "risk neutral" prices are recovered as special cases of our more general pricing theory when the claim being priced happens to be "replicable". Subjectivity is a fundamental property finance rather than a mathematical artefact. It explains why different agents might trade with each other and thus, why financial markets exist in the first place. Traditional models of financial mathematics have, unfortunately, downplayed the role of subjectivity and thus, missed much of what really matters in practice.

From now on, we assume the following.

Assumption 1 *$D \subseteq \mathbb{R}^J$ is a convex set containing the origin and $\mathcal{V} : L^0 \to \overline{\mathbb{R}}$ is nondecreasing convex function with $\mathcal{V}(0) = 0$.*

The condition $\mathcal{V}(0) = 0$ is posed mainly for notational convenience. As long as $0 \in \operatorname{dom} \mathcal{V}$, it can be achieved by adding a constant to the objective (note that adding a constant does not affect the agent's risk preferences nor problem (ALM)). Convexity corresponds to the fundamental diversification principle in risk management: the "risk" associated with a convex combination of two claims c_1 and c_2 should be no higher than the convex combination of the individual risks. The convexity of D and $\mathcal{V}$ imply that (ALM) is a convex optimization problem, i.e. a problem of minimizing a convex function over a convex set. Indeed, the objective is the *composition* of $\mathcal{V}$ with the affine function $x \mapsto c - s_1 \cdot x$ from $\mathbb{R}^J$ to L^0 so it is convex as a function of x; see Exercise 1. Convexity is an important property both for mathematical and numerical analysis of optimization problems.

The general pricing theory that will be developed in the following sections is be based on studying how the agent's financial position, as described by (w, c), affects the optimum value of (ALM). We will denote the optimum value by

$$\varphi(w, c) := \inf\{\mathcal{V}(c - s_1 \cdot x) \,|\, x \in D,\ s_0 \cdot x \leq w\}.$$

The following simple fact turns out to have important consequences in financial risk management and valuation of financial products.

Lemma 1.6 *The value function φ of* (ALM) *is nonincreasing in w, nondecreasing in c and $\varphi(0, 0) \leq 0$. Moreover, φ is convex.*

Proof. The first statement follows directly from the fact that $\mathcal{V}$ is nondecreasing and $\mathcal{V}(0) = 0$. The second statement is proved in Exercise 3. □

The convexity of φ may be seen as another manifestation of the diversification principle. In the following sections, we will see that the convexity of φ implies that values of contingent claims are convex functions of the claims. This fact allows us to bound the values between the so called "arbitrage bounds", which coincide with the classical replication based prices in complete market models. Essential to all this is the inclusion of the liability c in the optimization problem (ALM) and the dependence of the optimum value on c.

Exercises

Given a set C, its *indicator function* δ_C is the extended real-valued function defined by

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

1. Let X and U be linear spaces and show that
   (a) a set $C \subset X$ is convex if and only if δ_C is convex function.
   (b) if f_1, f_2 are convex functions on X, then $f(x) = f_1(x) + f_2(x)$ is a convex function on X. Here and in what follows, we define $\infty + (-\infty) = (-\infty) + \infty = \infty$.
   (c) if h is a convex function on U and $A : X \to U$ is linear, then $f(x) = h(Ax)$ is a convex function on X.
   (d) if $\{f_i\}_{i \in I}$ are convex functions on X, then $f(x) = \sup_{i \in I} f_i(x)$ is a convex function on X.
   (e) if f is a convex function on $X \times U$, then
   $$\varphi(u) = \inf_{x \in X} f(x, u)$$
   is convex on U.
2. Let X be a linear space and let C be a convex subset of $X \times \mathbb{R}$. Show that
   $$f(x) = \inf\{\alpha \mid (x, \alpha) \in C\}$$
   is a convex function on X. Use this to give a quick proof of (e) in Exercise 1.
3. Prove the convexity of φ as stated in Lemma 1.6.
4. Write problem (ALM) in terms of the decision variables
   (a) $h \in \mathbb{R}^J$, where h^j gives the amount of money invested in asset class $j \in J$.
   (b) $p \in \mathbb{R}^J$, where p^j gives the proportion of initial wealth w invested in asset j.

   Are the resulting problems convex?
5. Show that for a risk neutral agent with $\mathcal{V}(c) = Ec$, the optimum value $\varphi(w, c)$ is linear in c and that the optimal portfolios (ALM)
   (a) do not depend on c,
   (b) depend on s_1 only through its expectation Es_1.

   Assuming further that $D = \mathbb{R}^J_+$, what are the optimal portfolios?
6. Write problem (ALM) in terms of the decision variables $h^j = s_0^j x^j$ when $\mathcal{V}$ is the mean-variance criterion of Example 1.2.
7. Specify the constraint set D in order to model the following constraints on the portfolio.
   (a) At most 50% of initial wealth is invested in a given asset $k \in J$.

(b) The variance of the *portfolio return* $(s_1 \cdot x)/w$ is at most $\bar{\sigma}^2$ as viewed by a supervisor who believes that investment returns r follow a multivariate normal distribution with expectation $m \in \mathbb{R}^J$ and variance $V \in \mathbb{R}^{J\times J}$.

(c) The δ% Value at Risk of negative return $-(s_1 \cdot x)/w$ is less than l as viewed by the same supervisor as in part (b).

Are the resulting constraints convex?

8. Consider a financial market where the traded assets consist of cash, S&P 500 index and call and put options on the index with strike prices $K^1, \dots, K^n$ and one year to maturity. Specify the price vectors s_0 and s_1, where $t = 1$ is one year from now.

Further reading

Portfolio optimization has had a central role in mathematical finance ever since the pioneering work of Harry Markowitz [32]. Sharpe et al. [47] gives an excellent non-technical overview of optimal investment and financial markets. Much of the theory has been developed in a continuous-time setting, most famously by Merton [33]; see Rogers [45] for a recent account. In the literature, the focus is often on characterising optimal strategies, whereas in these notes, we focus on contingent claim valuation. This is why we have included the liabilities c in the optimization model (ALM) from the beginning. This is an important feature that has, unfortunately, been missing in most of the literature on optimal investment.

1.4 Accounting values

In financial reporting and supervision of financial institutions, one is often interested in determining the least amount of capital that would suffice for covering a financial liability at an acceptable level of risk. If the liability happens to be traded in liquid markets, one can simply use market prices for the valuation (such a valuation is often referred to as "marking to market"). In many situations, however, this is not possible so one has to detemine the required capital in another way. In most accounting standards, the aim is to come up with a value that reflects the current market conditions and relevant risks of having the liability.

In the single-period model of Section 1.3, where financial liabilities are described by random cash-flows $c \in L^0(\Omega, \mathcal{F}, P)$ at time $t = 1$, such an amount can be defined as

$$\pi_s^0(c) = \inf\{w \in \mathbb{R} \,|\, \varphi(w, c) \leq 0\}.$$

Indeed, $\pi_s^0(c)$ gives the least amount of initial capital w needed to construct a *hedging strategy* $x \in D$ whose value at time $t = 1$ covers a sufficient part of the liability c in the sense that the residual risk $\mathcal{V}(c - s_1 \cdot x)$ is no higher than the risk of doing nothing at all (recall that $\mathcal{V}(0) = 0$). Note that if $\varphi(w, c)$ is strictly decreasing in w, then $\varphi(w, c) = 0$ implies $\pi^0_s(c) = w$. If the liability c is liquidly traded in the market, $\pi^0_s(c)$ reduces to the market price of c; see Theorem 1.7 below. We will call $\pi^0_s(c)$ the *accounting value of a liability* $c \in L^0$.

If the optimum in (ALM) is attained for every w and c, we have

$$\pi^0_s(c) = \inf\{w \mid \exists x \in D,\ s_0 \cdot x \le w,\ \mathcal{V}(c - s_1 \cdot x) \le 0\}$$

so that $\pi^0_s(c)$ equals the optimum value of the convex optimization problem

$$\begin{array}{ll} \text{minimize} & s_0 \cdot x \quad \text{over} \quad x \in D, \\ \text{subject to} & \mathcal{V}(c - s_1 \cdot x) \le 0. \end{array} \tag{1.1}$$

If we ignore the existence of financial markets and assume that we can lend and borrow arbitrary amounts of cash at zero interest, we get

$$\pi^0_s(c) = \inf\{w \mid \mathcal{V}(c - w) \le 0\} \tag{1.2}$$

which is a risk measure in the sense of Example 1.4; see Exercise 3. This corresponds to classical premium principles in actuarial mathematics. By an appropriate choice of a trading strategy, however, one may be able to reduce the accounting value as compared to (1.2). The same idea is behind the Black–Scholes–Merton option pricing model where the price of an option is defined as the least amount of initial capital that allows for the construction of a trading strategy whose terminal value equals the payout of the option. In practice, however, exact replication is often impossible so it is important to specify preferences concerning the unhedged part $c - s_1 \cdot x$ of the claim. Our definition of the accounting value $\pi^0_s(c)$ unifies the actuarial and fiancial valuation principles in incomplete markets.

While $\pi^0_s(c)$ gives the least amount of initial capital required for constructing an acceptable (in the sense of the preferences given by $\mathcal{V}$) hedging strategy for a liability c, the number

$$\pi^0_l(c) := \sup\{w \in \mathbb{R} \mid \varphi(-w, -c) \le 0\}$$

gives the greatest amount of cash one could generate at time $t = 0$ by taking a position $x \in D$ (shorting some of the traded assets) that could be covered at an acceptable level of risk when receiving c units of cash at time $t = 1$. We will call $\pi^0_l(c)$ the *accounting value of an asset* $c \in L^0$. Clearly, $\pi^0_l(c) = -\pi^0_s(-c)$. Accounting values for assets are relevant e.g. in banks whose assets are divided into the *trading book* and the *banking book*. Assets in the banking book are typically illiquid assets such as individual loans without secondary markets. A bank usually holds such assets until maturity. Variable interest rates and credit risk make such a loan's cash flows c uncertain so the accounting value is not just the notional amount of the loan. Again, for liquidly traded assets, market prices should be used for valuation.

In order to clarify the connections between accounting values, market prices and risk-neutral valuation (as e.g. in the Black–Scholes–Merton model), consider the *superhedging cost*

$$\pi_{\sup}(c) := \inf\{w \mid \exists x \in D : \; s_0 \cdot x \le w, \; c \le s_1 \cdot x \; P\text{-a.s.}\}.$$

The superhedging cost gives the least amount of initial capital required to by a riskless hedging strategy for a claim $c \in L^0$. Analogously, the *subhedging cost*

$$\pi_{\inf}(c) := \sup\{w \mid \exists x \in D : \; s_0 \cdot x + w \le 0, \; c + s_1 \cdot x \ge 0 \; P\text{-a.s.}\}$$

gives the greatest amount of initial cash one could generate by entering a portfolio that could be risklessly covered when receiving c units of cash at time $t = 1$. Clearly, $\pi_{\inf}(c) = -\pi_{\sup}(-c)$.

Note that the super- and subhedging costs do not depend on the agent's risk preferences. Moreover, they depend on the probability measure P only through its null sets. Indeed, $\pi_{\sup}(c)$ does not change if P is replaced by any measure equivalent to P. The super- and subhedging costs are often called "arbitrage bounds" for the price of c. If it was possible to sell the claim c for more than $\pi_{\sup}(c)$, one could make riskless profit by selling the claim and following a superhedging strategy. Similar arbitrage opportunity would exist if we could buy the claim for less than $\pi_{\inf}(c)$.

The accounting value $\pi^0_s(c)$ defines an extended real-valued function on L^0. The following theorem summarizes some of its key properties. We will say that a claim $c \in L^0$ is *replicable* if there is an $x \in D$ such that $-x \in D$ and $s_1 \cdot x = c$ almost surely. Note that the condition $-x \in D$ holds automatically if there are no portfolio constraints, i.e. if $D = \mathbb{R}^J$.

Theorem 1.7 *The function π^0_s is convex, nondecreasing and $\pi^0_s(0) \le 0$. We always have $\pi^0_s(c) \le \pi_{\sup}(c)$. If $\pi^0_s(0) = 0$, then*

$$\pi_{\inf}(c) \le \pi^0_l(c) \le \pi^0_s(c) \le \pi_{\sup}(c) \quad \forall c \in L^0$$

with equalities throughout for replicable c.

Proof. The proof of convexity is left as an exercise. The function π^0_s is nondecreasing simply because $\mathcal{V}$ is. The fact that $\pi^0_s(0) \le 0$, follows from the fact that $\varphi(0,0) \le 0$ as noted in Lemma 1.6.

In order to prove $\pi^0_s(c) \le \pi_{\sup}(c)$, it suffices to note that if there exists an $\bar{x} \in D$ such that $s_0 \cdot \bar{x} \le w$ and $c \le s_1 \cdot \bar{x}$, then $\varphi(w,c) \le 0$ (look at the definitions of π^0_s and $\pi_{\sup}$). But this is clear since any such $\bar{x}$ is feasible in (ALM) so

$$\varphi(w,c) \le \mathcal{V}(c - s_1 \cdot \bar{x}) \le \mathcal{V}(0) = 0.$$

Since $\pi_{\inf}(c) = -\pi_{\sup}(-c)$ and $\pi^0_l(c) = -\pi^0_s(-c)$ we also have $\pi_{\inf}(c) \le \pi^0_l(c)$.

By convexity of π^0_s,

$$\pi^0_s(0) = \pi^0_s(\frac{1}{2}c + \frac{1}{2}(-c)) \le \frac{1}{2}\pi^0_s(c) + \frac{1}{2}\pi^0_s(-c)$$

so if $\pi_s^0(0) = 0$ we get $\pi_l^0(c) \le \pi_s^0(c)$, which completes the proof of the "sandwich" inequalities. Assume now that c is replicable with $s_1 \cdot \bar{x} = c$ almost surely for some $\bar{x} \in D$ with $-\bar{x} \in D$. Such an $\bar{x}$ superhedges c while $-\bar{x}$ subhedges c. Thus $\pi_{\sup}(c) \le s_0 \cdot \bar{x} \le \pi_{\inf}(c)$, which completes the proof. □

Convexity of the accounting value correspond to the classical diversification principle in risk management. The hedging cost of the convex combination of two claims is less than or equal to the convex combination of the individual hedging costs. Diversification benefits may result in a strict inequality so the hedging cost is in general a nonlinear function of the claims.

The accounting value π_s^0 behaves much like a *risk measure* in Example 1.4. Indeed, π_s^0 is convex and monotone but it does not have the translation property in general. While a risk measure gives the amount of cash required at time $t = 1$, the accounting value gives the required amount of cash at time $t = 0$. The accounting value does satisfy the translation property in Example 1.4 if arbitrary positions in cash are allowed; see Exercise 5. While this is a standard assumption in financial mathematics, it rarely holds in practice.

The condition $\pi_s^0(0) = 0$ means that the accounting value of the zero claim is zero. This is quite natural for an accounting value. It was used in the above proof to show that $\pi_l(c) \le \pi_s^0(c)$ for all claims $c \in L^0$. Violation of this inequality would mean that a company could improve its net accounting position simply by adding c at the same time to its assets and liabilities. Sensible accounting standards should not allow for such "accounting arbitrage opportunities". In terms of the optimum value function of (ALM), the condition $\pi_s^0(0) = 0$ means that $\varphi(w, 0) > 0$ when $w < 0$ (why?). Sufficient conditions for this are given in Exercise 4.

In general, π_s^0 depends on the the probability measure P and the function $\mathcal{V}$ (views and risk preferences) but the last part of Theorem 1.7 says that for replicable claims, π_s^0 is independent of such subjective factors (since the superhedging cost is so). In particular, accounting values of liquidly traded assets equal their market prices. Replicable claims are rarely found in practice so typically, accounting values do have subjective factors. In practice, $\mathcal{V}$ and P should be specified (or at least approved) by the supervisory authorities. There have been attempts to define accounting values that do not depend on subjective factors but in practice, such valuations only lead to confusions and financial instabilities.

We have assumed implicitly that one can solve problem (ALM) exactly. This is rarely the case in practice, so one has to resort to numerical techniques that can only generate approximate solutions and optimum values. When (1.1) cannot be solved exactly, the accounting value should be defined as the least value a financial institution *can* achieve in (1.1). It then depends also on the trading expertise of a financial institution. This is quite natural from the practical point of view: institutions that are good at hedging their liabilities should be allowed to operate with less capital.

Exercises

1. Prove the first claim of Theorem 1.7.

2. Consider a financial market with only one asset which can be traded without constraints and whose unit price is strictly positive in all scenarios. Show that for a risk neutral agent

$$\pi_s^0(c) = Ec/Er,$$

and, in particular, if the single asset is a zero coupon bond with price P_1, then one gets the discounted expectation $\pi_s^0(c) = P_1 Ec$.

3. Consider a financial market with only one asset which can be traded without constraints and whose unit price is constant one. Show that

   (a) $\pi_s^0(c)$ is a convex risk measure in the sense of Example 1.4.

   (b) if $\mathcal{V}(c) := E[e^{\gamma c} - 1]$, then

   $$\pi_s^0(c) = \frac{1}{\gamma} \ln E e^{\gamma c}.$$

   This function is known as the *entropic risk measure*.

   (c) if c_1 and c_2 are independent random variables, then the entropic risk measure satisfies

   $$\pi_s^0(c_1 + c_2) = \pi_s^0(c_1) + \pi_s^0(c_2).$$

4. Show that the condition $\pi_s^0(0) = 0$ in Theorem 1.7 holds under either of the following conditions

   (a) $D \subseteq \mathbb{R}^J_+$ and $s_0 \in \mathbb{R}^J_+$.

   (b) $\operatorname{dom} \mathcal{V} \subseteq L^0_-$ and the market satisfies the *weak no-arbitrage* condition: there is no $x \in D$ such that

   $$s_0 \cdot x < 0 \quad \text{and} \quad s_1 \cdot x \geq 0 \ P\text{-a.s.}.$$

5. In financial mathematics it is often assumed that there is a special asset (e.g. a bank account), say $0 \in J$, whose unit price s^0 is always strictly positive and which can be traded without constraints. Although, this is not quite a realistic assumption, it can nevertheless be modeled in our framework by setting $s_t = (s_t^0, \tilde{s}_t)$, where $\tilde{s}_t$ denotes the prices of the "risky" assets $\tilde{J} = J \setminus \{0\}$, and $D = \mathbb{R} \times \tilde{D}$, where $\tilde{D} \subseteq \mathbb{R}^{\tilde{J}}$ is a convex set. Expressing all prices in terms of asset 0, one then has that $s_t^0 \equiv 1$ (asset 0 is then said to be the *numeraire*). Show that in this case

   (a) $\varphi(w, c + \alpha) = \varphi(w - \alpha, c)$ for all $\alpha \in \mathbb{R}$.

   (b) $\pi_s^0(c + \alpha) = \pi_s^0(c) + \alpha$ for all $\alpha \in \mathbb{R}$.

(c) If $\pi^0_s(0) = 0$, then π^0_s is a convex risk measure in the sense of Example 1.4.

6. While the assumptions of Exercise 5 are unrealistic, it is sometimes reasonable to assume that one can invest arbitrary positive amounts in cash ("putting money under mattress"). This can be modeled by setting $s_t = (1, \tilde{s}_t)$ and $D = \{(x^0, \tilde{x}) \,|\, x^0 \geq 0,\ \tilde{x} \in \tilde{D}\}$. Show that in this case

   (a) $\varphi(w, c + \alpha) \leq \varphi(w - \alpha, c)$ for all $\alpha \geq 0$.

   (b) $\pi^0_s(c + \alpha) \leq \pi^0_s(c) + \alpha$ for all $\alpha \geq 0$.

7. Assume that $\mathcal{R} : L^\infty \to \mathbb{R}$ satisfies properties 1 and 2 of Example 1.4 but instead of 3 it only satisfies $\mathcal{R}(\alpha) = \alpha$ for every $\alpha \in \mathbb{R}$. Show that $\mathcal{R}$ is a risk measure.

8. If $\mathcal{R}$ is a convex risk measure in the sense of Example 1.4, then the function

$$\mathcal{V}(c) = \inf_{d \in L^\infty} \{\mathcal{R}(d) \,|\, d \geq c\}$$

   is a convex nondecreasing function on L^0.

Further reading

Our formulation of the accounting value π^0_s corresponds to the usual definition of an "option price" as the least amount of initial cash needed for hedging the option. The most famous example is the classical Black–Scholes–Merton model where essentially all claims are *replicable* ("attainable" in the terminology of [17]) by appropriate trading; see [4]. When replication is not possible, the terminal wealth is necessarily uncertain and one has to specify what kind of terminal positions are acceptable. The totally risk averse attitude leads to superhedging. Comprehensive treatments of superhedging in incomplete markets can be found in [17, 12, 26]. More reasonable risk preferences are studied in a liquid multiperiod model in Föllmer and Schied [17, Chapter 8]. Illiquid generalizations can be found e.g. in Davis, Panas and Zariphopoulou [11] and Section 2.3 below. Expressions of the form (1.2) are often used in actuarial mathematics as premium principles for insurance claims; see e.g. Bühlmann [5].

Our formulation of (1.1) is also motivated by modern financial supervisory standards, such as the Solvency II Directive 2009/138/EC of the European Parliament, which promotes market consistent accounting principles that recognize the risks in both assets and liabilities. The interplay between accounting values and asset management was recently studied in Artzner, Delbaen and Koch-Medona [2]. An implementation of a dynamic version of (1.1) (see Section 2.3 below) for pension insurance liabilities is described in Hilli, Koivu and Pennanen [20].

1.5 Indifference pricing

Consider now the problem of valuing a contingent claim $c \in L^0(\Omega, \mathcal{F}, P)$ from the point of view of an agent whose current financial position is given by an initial wealth $\bar{w} \in \mathbb{R}$ and liabilities $\bar{c} \in L^0(\Omega, \mathcal{F}, P)$. This time we are not looking for the amount of capital that allows the agent to achieve acceptable level of risk with a given liability but a price at which he could sell/buy a claim c without worsening his financial position as measured by the optimum value of (ALM). One would expect that such a price depends on the agent's financial position before the trade. For instance, the price you would be willing to pay for a car insurance probably depends on whether you own a car or not. Similarly, a wheat farmer is more likely to buy a futures contract on wheat than somebody who's income does not depend on wheat price. In fact, most financial instruments have been created because of differences in financial positions of different agents. Appropriately designed financial contracts allow both counterparties of a trade to improve their financial position. This is yet another fundamental principle of finance that cannot be explained by complete market models.

Consider an agent whose financial position is described by $(\bar{w}, \bar{c}) \in \mathbb{R} \times L^0$ in the model of Section 1.3. The least cash-payment at time $t = 0$ that would compensate the agent for delivering a claim c at time $t = 1$ can be expressed as

$$\pi_s(\bar{w}, \bar{c}; c) = \inf\{w \mid \varphi(\bar{w} + w, \bar{c} + c) \leq \varphi(\bar{w}, \bar{c})\}, \tag{1.3}$$

where again, $\varphi(w, c)$ denotes the optimum value of (ALM). The value $\pi_s(\bar{w}, \bar{c}; c)$ is called the *indifference selling price* of c. It gives the least price at which the agent would be willing to sell the claim c. Any price $w > \pi_s(\bar{w}, \bar{c}; c)$ would allow the agent to reoptimize his portfolio after the trade so that the risk with his new financial position $(\bar{w} + w, \bar{c} + c)$ is no greater than it was before the trade. Note that if $\varphi(w, c)$ is strictly decreasing in w, then $\varphi(\bar{w} + w, \bar{c} + c) = \varphi(\bar{w}, \bar{c})$ implies $\pi_s(\bar{w}, \bar{c}; c) = w$.

While π_s gives the least price an agent would be willing to sell a claim for, the *indifference buying price* for c is given by

$$\pi_l(\bar{w}, \bar{c}; c) = \sup\{w \mid \varphi(\bar{w} - w, \bar{c} - c) \leq \varphi(\bar{w}, \bar{c})\}.$$

It answers the question: "How much would you be willing to pay to reduce your liabilities by c?". Such questions are relevant when buying e.g. insurance products. Clearly, $\pi_l(\bar{w}, \bar{c}; c) = -\pi_s(\bar{w}, \bar{c}; -c)$.

The indifference prices are not necessarily the prices at which one would actually trade in pratice. Financial institutions and individuals usually add a "margin" on top of these prices in order to cover operational expenses and to make profit. The above indifference prices should be understood as *bounds* on what a rational agent would offer.

Under a mild condition, the indifference selling price dominates the indifference buying price, as we will see in the theorem below. Adding margins would increase this spread. When there are no constraints, i.e. when $D = \mathbb{R}^J$, the indifference selling price is bounded from above by the superhedging cost just like the accounting value in Section 1.4. A generalization to a constrained setting will be given in Section 2.4.

Theorem 1.8 *The function $\pi_s(\bar{w},\bar{c};\cdot)$ is convex, nondecreasing and $\pi_s(\bar{w},\bar{c};0) \le 0$. If there are no portfolio constraints, then $\pi_s(\bar{w},\bar{c};c) \le \pi_{\sup}(c)$. If in addition, $\pi_s(\bar{w},\bar{c};0) = 0$, then*

$$\pi_{\inf}(c) \le \pi_l(\bar{w},\bar{c};c) \le \pi_s(\bar{w},\bar{c};c) \le \pi_{\sup}(c) \quad \forall c \in L^0$$

with equalities throughout when c is replicable.

Proof. Since, by Lemma 1.6, $\varphi(w,c)$ is nondecreasing in c, the constraint

$$\varphi(\bar{w}+w,\bar{c}+c) \le \varphi(\bar{w},\bar{c})$$

becomes more restrictive on w when c increases. Thus, if c increases, the infimum in the definition of $\pi_s(\bar{w},\bar{c};c)$ increases if anything. The fact that $\pi_s(\bar{w},\bar{c};0) \le 0$ is also clear from the definition of π_s. The proof of convexity is left as an exercise.

For any $w > \pi_{\sup}(c)$, there is an $x' \in \mathbb{R}^J$ such that $s_0 \cdot x' \le w$ and $c \le s_1 \cdot x'$ almost surely. When there are no constraints, the change of variables $x \to x - x'$ in (ALM) gives

$$\varphi(\bar{w},\bar{c}) = \varphi(\bar{w}+s_0\cdot x',\bar{c}+s_1\cdot x').$$

The growth properties of φ in Lemma 1.6 then give

$$\varphi(\bar{w},\bar{c}) \ge \varphi(\bar{w}+w,\bar{c}+c)$$

and thus, $\pi_s(\bar{w},\bar{c};c) \le w$. Since $w > \pi_{\sup}(c)$ was arbitrary, we must have $\pi_s(\bar{w},\bar{c};c) \le \pi_{\sup}(c)$.

By convexity of $\pi_s(\bar{w},\bar{c};\cdot)$,

$$\pi_s(\bar{w},\bar{c};0) \le \frac{1}{2}\pi_s(\bar{w},\bar{c};c) + \frac{1}{2}\pi_s(\bar{w},\bar{c};-c)$$

so if $\pi_s(\bar{w},\bar{c};0) = 0$, we get $-\pi_s(\bar{w},\bar{c};-c) \le \pi_s(\bar{w},\bar{c};c)$. The "sandwich" inequalities now come from the identities $\pi_l(\bar{w},\bar{c};c) = -\pi_s(\bar{w},\bar{c};-c)$ and $\pi_{\inf}(c) = -\pi_{\sup}(-c)$. The last claim follows from the last claim of Theorem 1.7. □

The condition $\pi_s(\bar{w},\bar{c};0) = 0$ means that one cannot lower the initial wealth without increasing the optimum value of (ALM). It holds, in particular, if $\varphi(w,c)$ is strictly decreasing in the initial endowment w. The condition was used to prove the inequality $\pi_l(\bar{w},\bar{c};c) \le \pi_s(\bar{w},\bar{c};c)$ which just means that it does not make sense for an agent to buy and sell the same product at the same time, or equivalently, two identical agents have no incentive to trade with each other.

Like accounting values, the indifference prices depend on an agent's future views as described by P and risk preferences given by $\mathcal{V}$. The indifference prices depend also on an agent's *financial position* $(\bar{w},\bar{c})$. All the three factors are subjective. Differences in any of them may give incentives for trading. In other words, it may happen that one agent's selling price $\pi_s(\bar{w}, \bar{c}; c)$ is strictly less than another agent's buying price $\pi_l(\bar{w}, \bar{c}; c)$. In fact, the subjectivity of valuations is the driving force behind trading and financial markets. Actively traded financial products are characterized by the existence of large numbers of agents with roughly opposite "exposures" to the cash-flows c of the traded product. In other words, there are several agents whose $\bar{c}$ tends to move in the same direction with c while for other agent's $\bar{c}$ tends to move in the opposite direction (think for example of a weat farmer and a baker, whose future cash-flows depend on the weat price).

Traditional complete market models do not explain why trading occurs in practice. Indeed, the second part of Theorem 1.8 says that for *replicable* claims, the buying and selling prices are equal and given by the superhedging cost $\pi_{\sup}$ which is independent of subjective factors. Thus, in complete market models, all agents would have the same buying and selling prices. Mere equality is not enough to trigger trading since, in practice, agents require nonzero margins to enter a trade. Note also that the convexity of $\pi_{\sup}$ and the concavity of $\pi_{\inf}$ imply that, on the space of replicable claims, prices are linear in c (since a function which is both convex and concave is necessarily "affine", i.e. linear plus a constant). In general, the selling price $\pi_s(\bar{w}, \bar{c}; \cdot)$ is a nonlinear convex function while the buying price $\pi_l(\bar{w}, \bar{c}; \cdot)$ is nonlinear and concave.

Combining Theorems 1.7 and 1.8, we get the following.

Corollary 1.9 *In unconstrained market models with $\pi_s^0(0) = 0$ and $\pi_s(\bar{w}, \bar{c}; 0) = 0$, accounting values and indifference prices of replicable claims are all equal to the superhedging cost.*

Thus, in complete market models (models where all claims can be replicated) such as the classical Black–Scholes model, there is no need to distinguish between accounting values and indifference prices. In practice, however, accounting values may be very be different from offered buying and selling prices (see Exercise 3) while the superhedging cost often takes the value $+\infty$.

The pricing principle (1.3) assumes that one can solve (ALM) to optimality. This assumption rarely holds in practice. However, (1.3) still makes sense if one redefines φ as the least value one *can* achieve in (ALM). Besides the financial position $(\bar{w}, \bar{c})$, future views P and risk preferences $\mathcal{V}$, offered prices then depend on an agents' trading expertise in optimizing their portfolios.

Exercises

1. Prove the convexity of $\pi_s(\bar{w}, \bar{c}; \cdot)$ claimed in Theorem 1.8.

2. Consider the single-asset setting of Exercise 2 of Section 1.4. Show that for a risk neutral agent, $\pi_s(\bar{w}, \bar{c}; c) = \pi_s^0(c)$.

3. Consider the single-asset setting of Exercise 3 of Section 1.4 and show that

   (a) $\pi_s(\bar{w}, \bar{c}; c + \alpha) = \pi_s(\bar{w}, \bar{c}; c) + \alpha$.

(b) if $\mathcal{V}(c) = E[e^{\gamma c} - 1]$ then $\pi_s(\bar{w}, \bar{c}; c)$ is independent of the initial wealth $\bar{w} \in \mathbb{R}$ and

$$\pi_s(\bar{w}, \bar{c}; c) = \frac{1}{\gamma} \ln \frac{Ee^{\gamma(\bar{c}+c)}}{Ee^{\gamma \bar{c}}}.$$

(c) if c and $\bar{c}$ are independent random variables, then $\pi_s(\bar{w}, \bar{c}; c) = \pi_s^0(c)$.

(d) if $\bar{c} = -c$, then $\pi_s(\bar{w}, \bar{c}; c) = \pi_l^0(c)$.

4. Consider the single-asset setting of Exercise 3 of Section 1.4. Show that the certainty equivalent in Exercise 3 of Section 1.3 can be expressed as $CE(c) = \pi_l(0, c; c)$. Thus, $\pi_l(0, c; c)$ may be viewed as a generalization of the certainty equivalent to nontrivial market models. Our definition of the indifference buying price $\pi_l(\bar{w}, \bar{c}; c)$ is still more general since it allows for an arbitrary financial position $(\bar{w}, \bar{c})$.

Further reading

Indifference pricing goes back, at least, to Hodges and Neuberger [21] who studied the pricing of contingent claims that cannot be exactly replicated due to transaction costs; see Carmona [6] for further references. Indifference pricing is a popular approach for pricing contingent claims that cannot be exactly replicated by trading in financial markets. Examples include various insurance contracts and credit derivatives; see Bühlmann [5] and Bielecki et al. [3], respectively. We will extend indifference pricing to general swap contracts in a multiperiod setting in Section 2.4 below.

1.6 Transaction costs and illiquidity

The market model considered so far describes *liquid* markets where unit prices of securities do not depend on whether one is buying or selling nor on the quantity of the traded amount. In real securities markets based on auction mechanisms, however, the unit price for buying is usually strictly higher than the unit price for selling. The best available price for buying is known as the *ask-price* while the best available price for selling is known as the the *bid-price*. Their difference is called the *bid-ask spread*. Proportional transaction costs can also be described in terms of the bid-ask spread. Indeed, transaction costs defined as a fixed proportion δ of the traded amount effectively increase the ask-price by a multiple of $(1 + \delta)$ and decrease the bid-price by a multiple of $(1 - \delta)$.

The numbers of securities that are available at the bid- and ask-prices are always finite. Trading larger quantities is usually possible if one is willing to accept worse prices (higher prices when buying and lower prices when selling). That is, when the traded quantities increase, prices move against us. Thus, unit prices of traded securities are *nonlinear* functions of our trades; they depend not only on the sign (buy or sell) but also on the quantity of a trade. The nonlinearity is generally referred to as *illiquidity*. Illiquidity increases trading costs and thus interferes with optimal investment and the valuation and hedging of contingent claims. Illiquidity is a sourse of incompleteness in practice.

Many securities are traded in *limit order markets*, where market participants submit buying or selling offers defined by limits on quantity and unit price. For example, a selling offer consists of an offer to sell up to x units of a security at the unit price of p (units of cash). At the beginning of a trading day, all available selling offers are combined into a function $x \mapsto s(x)$ called the *supply curve*. It gives the *marginal price* for buying x units among the best available selling offers. The quantity available at the selling price is finite and when buying more one gets the second lowest price and so on. The supply curve s is thus be a piecewise constant nondecreasing function of the amount bought. Buying offers are combined analogously into a piecewise constant nonincreasing function d known as the *demand curve*. It gives the marginal price for selling as a function of the number of units sold among the best available buying offers.

The market is cleared by matching the maximum amount of selling orders and buying orders by finding the largest number $\bar{x}$ such that $s(\bar{x}) \leq d(\bar{x})$. In other words, there is exactly $\bar{x}$ units of buying offers whose unit prices exceed or match the unit prices of $\bar{x}$ units of selling offers. The prices of the remaining selling offers are then all strictly higher than the prices of the remaining buying offers and no more trades are possible before new offers arrive. The *market clearing price* is a price in the interval $[s(\bar{x}), d(\bar{x})]$.

The above describes what happens in a so called *call auction* that takes place at the beginning of a trading day before continuous trading begins. The offers remaining after market clearing are recorded in the so called *limit order book*. It gives the marginal prices for buying or selling a given quantity at the best available prices. Interpreting negative purchases as sales, the marginal prices can be incorporated into a single function $x \mapsto s(x)$ giving the marginal price for buying positive or negative quantities of the asset. Since the highest buying price is lower than the lowest selling price, the marginal price curve s is a nondecreasing piecewise constant function on $\mathbb{R}$. Figure 1.6 presents an example of a marginal price curve s taken from Copenhagen stock exchange.

The function

$$S(x) = \int_0^x s(z)dz$$

gives the *cost* of buying x shares after market clearing. It is a piecewise linear convex function on $\mathbb{R}$; see Exercise 1 in Section 1.2. Again, negative x is interpreted as sales and a negative cost as revenue. Note that the perfectly liquid market model studied earlier corresponds to marginal prices $s(x)$ being independent of the traded quantity x in which case the cost function S is linear. Proportional transaction costs would result in a positively homogeneous convex cost function. Sections 1.3–1.5 could be readily extended to allow for illiquidity effects. We will do this in the next section in a dynamic setting.

What we have explained above, applies to so called *market orders*. In a market order, a trader buys a desired number x of securities at best available prices. Besides market orders, market participants may also submit *limit orders* which do not lead to an immediate transaction if the offered price does not

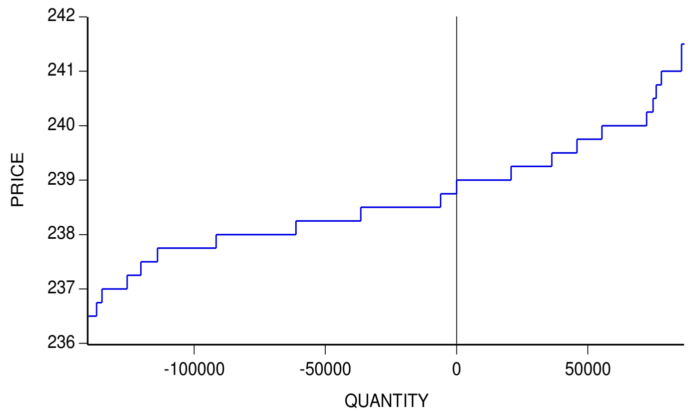

Figure 1.1: Marginal price curve for shares of the Danish telecom company TDC A/S observed in Copenhagen Stock Exchange on January 12, 2005 at 13:58:19.43. The horizontal axis gives the cumulative depth of the book measured in the number of shares. Negative order quantity corresponds to a sale. The prices are in Danish krone. The data was provided by OMX market research.

match available prices in the book. Such orders are incorporated into the limit order book. While market orders reduce liquidity by truncating the limit order book, limit orders increase liquidity.

Exercises

Given an extended real-valued function f on $\mathbb{R}^J$, a vector $v \in \mathbb{R}^J$ is said to be a *subgradient* of f at $\bar{x} \in \mathbb{R}^J$ if

$$f(x) \geq f(\bar{x}) + v \cdot (x - \bar{x}) \quad \forall x \in \mathbb{R}^J.$$

The set of subgradients of f at $\bar{x}$ is known as the *subdifferential* of f at $\bar{x}$ and it is denoted by $\partial f(\bar{x})$. Being the intersection of closed half-spaces, $\partial f(\bar{x})$ is a closed convex set. It can be shown that if f is differentiable at $\bar{x}$, then $\partial f(\bar{x})$ reduces to a single point, the gradient of f at $\bar{x}$. Clearly, an $\bar{x}$ minimizes f if and only if $0 \in \partial f(\bar{x})$. This generalizes the classical *Fermat's rule* which says that the gradient of a differentiable function vanishes at a minimizing point. In optimization, the classical notions of differentiability and the gradient are often replaced by convexity and the subgradient.

1. Let f be a convex function on $\mathbb{R}$. Show that ∂f is a nondecreasing curve.

2. Let f and $\bar{x}$ be as in Exercise 1 in Section 1.2. Show that $\partial f(\bar{x})$ is the closed interval between the left and right limits of ϕ at $\bar{x}$.

3. Let f be a function on an interval $I \subseteq \mathbb{R}$. Prove that if $f \in C^2$ with $f'' \geq 0$, then f is convex.

4. Let f be a convex function on an interval $I \subseteq \mathbb{R}$. Prove that f is differentiable on the interior $\operatorname{int} I$ of I except perhaps on a countable set, its derivative f' is nondecreasing and

$$f(x) = f(\bar{x}) + \int_{\bar{x}}^{x} f'(z)dz$$

for all $\bar{x}, x \in \operatorname{int} I$.

Further reading

A general, nontechnical account of various trading protocols used in securities markets can be found in Harris [18]. Limit order markets have attracted a lot of attention in mathematical finance in recent years. The multivariate characteristics of limit order books provide many challenges in statistical modeling of financial markets. A simple parametric approach is presented in Malo and Pennanen [31].

Another generalization of the classical linear model of financial markets was proposed in Kabanov [25]; see also Kabanov and Safarian [26]. In Kabanov's model, all assets are treated symmetrically, much like in currency markets, and budget constraints are described in terms of "solvency cones", which can be interpreted as the negatives of the sets of portfolios that are freely available in the market. In the model of limit order markets, such a set can be expressed as $\{x \in \mathbb{R}^J \mid S(x) \leq 0\}$. In Kabanov's original model the sets were polyhedral cones. Extensions to more general convex sets were studied in Pennanen and Penner [39].

An extensive treatment of differential properties of convex functions on finite-dimensional spaces can be found in Rockafellar [40, Part V].

Chapter 2

Dynamic models

This chapter extends the incomplete markets theory to a dynamic setting where financial products may have several payment dates and investment portfolios may be updated when payments are made or when new information becomes available. The theory will be developed in a general market model that allows for illiquidity effects in the form of nonlinear trading costs and portfolio constraints. Such models are needed to explain certain features of financial markets that are not captured by single-period models or the traditional models of financial mathematics.

Much of trading in practice involves exchanging sequences of cash-flows. Examples include swaps, coupon paying bonds, divident paying stocks and most insurance contracts where claims as well as premiums may have several payment dates. Such contracts do not fit traditional models of financial mathematics which assumes the existence of a perfectly liquid cash-account that allows one to postpone all payments until a single maturity date. If such an asset existed, swaps, dividents and coupon payments would be redundant. In practice, however, the timing of payments is an important issue. This became painfully obvious during the financial crisis of 2008 when the lack of credit and liquidity caused trouble to financial institutions and individuals around the world.

This chapter extends the classical pricing theory to financial contracts with several payment dates and to financial markets with illiquidity effects in terms of nonlinear trading costs and portfolio constraints. Allowing for dynamic portfolio updating leads to infinite-dimensional optimization problems where trading strategies are described by sequences of random vectors. This is a significant complication compared to single-period models where portfolios are described by finite-dimensional vectors. Fortunately, the convex analytic approach of the previous sections extends easily to infite-dimensional spaces of trading strategies. Convex analysis also suggests how to extend the basic notions of arbitrage bounds and replicable claims to nonlinear models with illiquidity effects. As we will see, the results of the previous sections can be derived from the more general results of this section as simple special cases.

2.1 Market model

We continue to model uncertainty by a general probability space $(\Omega, \mathcal{F}, P)$ but now one may trade at multiple points $t = 0, \dots, T$ in time. The information available at time t is described by a σ-algebra $\mathcal{F}_t \subseteq \mathcal{F}$ in the sense that, at time t, we do not know which scenario ω will eventually realize but only which element of $\mathcal{F}_t$ it belongs to. We will assume that the sequence $(\mathcal{F}_t)_{t=0}^T$ is a *filtration*, i.e. $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$. This just means that the information increases over time.

A portfolio x_t chosen at time t may depend on the information observed so far but not on information that will only be observed in the future. This means that x_t is an $\mathcal{F}_t$-measurable function from Ω to $\mathbb{R}^J$, or in other words, the trading strategy $x = (x_t)_{t=0}^T$ is *adapted*[1] to the filtration $(\mathcal{F}_t)_{t=0}^T$. The linear space of adapted trading strategies will be denoted by $\mathcal{N}$. Unless $\mathcal{F}_T$ has only a finite number of elements with positive probability, $\mathcal{N}$ is an infinite-dimensional space. We will assume that $\mathcal{F}_0 = \{\Omega, \emptyset\}$ so that x_0 is independent of ω. Following a trading strategy $x \in \mathcal{N}$ requires buying a portfolio $\Delta x_t := x_t - x_{t-1}$ at time t. Again, negative purchases are interpreted as sales.

We will denote the cost of buying a portfolio $z \in \mathbb{R}^J$ at time t in scenario ω by $S_t(z, \omega)$. We assume that, for each $t = 0, \dots, T$ the function S_t is $\mathcal{B}(\mathbb{R}^J) \otimes \mathcal{F}_t$-measurable and that $z \mapsto S_t(z, \omega)$ is *convex* and vanishes at the origin. The financial market is thus described by an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence $(S_t)_{t=0}^T$ of random convex functions that vanish at the origin. Such a sequence is called a *convex cost process.*

The convexity of $S_t(\cdot, \omega)$ was motivated in Section 1.6 but note that now, we are considering the cost of a portfolio instead of a single asset; see the examples below. Following a trading strategy $x \in \mathcal{N}$ requires investing $S_t(\Delta x_t)$ units of cash at time t. The measurability of S_t implies that $\omega \mapsto S_t(\Delta x_t(\omega), \omega)$ is $\mathcal{F}_t$-measurable for each t, i.e. the cost of a portfolio is known at the time of purchase. Indeed, the cost is the composition of the measurable functions $\omega \mapsto (\Delta x_t(\omega), \omega)$ and S_t.

Example 2.1 *If s_t is an $\mathbb{R}^J$-valued $\mathcal{F}_t$-measurable price vector for each $t = 0, \dots, T$, then the functions*

$$S_t(x, \omega) = s_t(\omega) \cdot x$$

define a linear cost process. This corresponds to a frictionless market, where unit prices are independent of purchases.

Example 2.2 *If $\bar{s}_t$ and $\underline{s}_t$ are $\mathbb{R}^J$-valued $\mathcal{F}_t$-measurable price vectors with $\underline{s}_t \leq \bar{s}_t$, then the functions*

$$S_t(x, \omega) = \sum_{j \in J} S_t^j(x^j, \omega),$$

[1] Some authors describe trading strategies by "predictable" processes $\xi = (\xi_t)_{t=0}^T$ with ξ_t denoting the portfolio that was chosen at time $t-1$. This is only a notational difference with $\xi_t = x_{t-1}$.

where

$$S_t^j(x^j,\omega) = \begin{cases} \overline{s}_t^j(\omega)x^j & \text{if } x^j \geq 0,\\ \underline{s}_t^j(\omega)x^j & \text{if } x^j \leq 0\end{cases}$$

define a sublinear *(i.e. convex and positively homogeneous) cost process. This corresponds to a market with transaction costs or bid-ask spreads, where unlimited amounts of all assets can be bought or sold for prices $\overline{s}_t$ and $\underline{s}_t$, respectively. When $\underline{s} = \overline{s}$, we recover Example 2.1.*

Example 2.3 *If Z_t is an $\mathcal{F}_t$-measurable set-valued mapping*[2] *from Ω to $\mathbb{R}^J$, then the functions*

$$S_t(x,\omega) = \sup_{s\in Z_t(\omega)} s\cdot x$$

define a sublinear cost process. When $Z_t = [\underline{s}_t, \overline{s}_t]$ we recover Example 2.2.

Proof. Sublinearity is left as an exercise. By [44, Example 14.51], $S_t(x,\omega)$ is an $\mathcal{F}_t$-measurable normal integrand[3]. By [44, Corollary 14.34], this implies that S_t is $\mathcal{B}(\mathbb{R}^J)\otimes\mathcal{F}_t$-measurable. □

We will model *portfolio constraints* by requiring that the portfolio $x_t(\omega)$ chosen at time t and state ω belongs to a given set $D_t(\omega)$. We will assume that $\omega \mapsto D_t(\omega)$ is $\mathcal{F}_t$-measurable closed convex-valued and that $0 \in D_t$ almost surely. The measurability condition simply means that the set D_t of feasible portfolios is known to us at time t when we choose x_t. The condition $0 \in D_t$ means that we are allowed not to hold any of the traded assets.

Besides short selling constraints, which correspond to $D_t(\omega) \equiv \mathbb{R}^J_+$, the portfolio constraints can be used to model situations where one encounters different interest rates for lending and borrowing; see Exercise 5. One can use D also to describe regulatory reguirements much like in the single-period seting in Exercise 7 of Section 1.3.

Exercises

A function f on a linear space X is said to be *sublinear* if it is convex and *positively homogeneous* in the sense that $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and $\alpha > 0$.

1. Show that the functions $S_t(\cdot,\omega)$ in Example 2.3 are sublinear.

2. Show that

   (a) a function f is sublinear if and only if

   $$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

   for every $x_i \in \operatorname{dom} f$ and $\alpha_i > 0$.

[2] A set-valued mapping $\omega \mapsto C(\omega)$ is $\mathcal{F}_t$-measurable if $\{\omega\in\Omega \mid C(\omega)\cap U \neq \emptyset\} \in \mathcal{F}_t$ for every open set U.

[3] A function $f : \mathbb{R}^J \times \Omega \to \overline{\mathbb{R}}$ is an *$\mathcal{F}_t$-measurable normal integrand* if $f(\cdot,\omega)$ is lower semicontinuous for every $\omega\in\Omega$ and if $\omega \mapsto \operatorname{epi}(\cdot,\omega)$ is an $\mathcal{F}_t$-measurable set-valued mapping.

(b) a positively homogeneous function f is convex if and only if

$$f(x_1 + x_2) \leq f(x_1) + f(x_2)$$

for every $x_i \in \operatorname{dom} f$.

3. The *mark-to-market value* of a portfolio $x \in \mathbb{R}^J$ is given by $s \cdot x$, where $s \in \mathbb{R}^J$ is a vector of *market prices*. At time t, market prices are given by an $\mathcal{F}_t$-measurable vector s_t with $s_t(\omega) \in \partial S_t(0, \omega)$, i.e.

$$S_t(z, \omega) \geq S_t(0, \omega) + s_t(\omega) \cdot z \quad \forall z \in \mathbb{R}^J.$$

Assume that s_t are componentwise strictly positive and express the trading cost $S_t(\Delta x_t)$ in terms of the mark-to-market values $h_t^j = s_t^j x_t^j$ of the investments x_t.

4. Specify a market model (S, D) in a situation where there is a perfectly liquid bank account.

5. Specify a market model (S, D) that describes a situation where the interest rates for lending and borrowing unlimited amounts of cash over each period $(t, t+1]$ are given by r_t^+ and r_t^-, respectively.

Further reading

Classical references on financial market models with a proportional transaction costs include Hodges and Neuberger [21], Jouini and Kallal [24] and Cvitanić and Karatzas [9]. Models with nonlinear cost functions have been studied in Çetin and Rogers [7] and Pennanen [35]. For a mathematical study of general measurable set-valued mappings and normal integrands, we refer the reader to Rockafellar [42] and Rockafellar and Wets [44, Chapter 14].

2.2 Asset-liability management

Consider an agent whose financial position is described by an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence $c = (c_t)_{t=0}^T$ of cash-flows in the sense that the agent has to deliver a random amount c_t of cash at time t. Allowing c to take both positive and negative values, endowments and liabilities can be modeled in a unified manner. In particular, $-c_0$ may be interpreted as an initial endowment while the subsequent payments c_t, $t = 1, \dots, T$ may be interpreted as the cash-flows associated with financial liabilities. A typical example would be an insurance portfolio that may obligate an insurer to claim payments over several points in time. Simpler claims such as European options correspond to processes c with $c_t = 0$ for all but one t. We will denote the space of $(\mathcal{F}_t)_{t=0}^T$-adapted sequence of cash-flows by

$$\mathcal{M} := \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P)\}.$$

The single-period ALM problem from Section 1.3 can be generalized to the dynamic setting as follows

$$\text{minimize} \quad \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t) \quad \text{over} \quad x \in \mathcal{N}_D, \tag{ALM-d}$$

where $\mathcal{V}_t$ are nondecreasing, convex functions on $L^0(\Omega, \mathcal{F}_t, P)$ and

$$\mathcal{N}_D := \{x \in \mathcal{N} \mid x_t \in D_t,\ t = 0, \ldots, T-1,\ x_T = 0\}.$$

We always define $x_{-1} = 0$ so the elements of $\mathcal{N}_D$ describe trading strategies that start and end at liquidated positions. We allow $\mathcal{V}_t$ to be extended real-valued but assume that $\mathcal{V}_t(0) = 0$.

Problem (ALM) can be interpreted as an *asset-liability management* problem where one looks for a trading strategy $x \in \mathcal{N}_D$ whose proceeds fit the given liabilities $c \in \mathcal{M}$ as well as possible. The functions $\mathcal{V}_t$ measure the disutility caused by the net expenditure $S_t(\Delta x_t)+c_t$ of updating the portfolio and paying out the claim c_t at time t. When $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$, we can write (ALM-d) with explicit constraints as

$$\begin{aligned} &\text{minimize} && \mathcal{V}_T(S_T(\Delta x_T) + c_T) \quad \text{over} \quad x \in \mathcal{N}_D, \\ &\text{subject to} && S_t(\Delta x_t) + c_t \le 0, \quad t = 0, \ldots, T-1. \end{aligned} \tag{2.1}$$

The constraint means that the claim c_t has to be completely financed by trading in financial markets. Recall that both the claim and the trading cost may be negative. When $T = 1$ and $S_t(x, \omega) = s_t(\omega) \cdot x$, we recover the single-period problem (ALM) from Section 1.3.

Example 2.4 (Single-period models) *When $T = 1$, problem* (2.1) *becomes*

$$\begin{aligned} &\text{minimize} && \mathcal{V}_1(S_1(\Delta x_1) + c_1) \quad \text{over} \quad x \in \mathcal{N}_D, \\ &\text{subject to} && S_0(\Delta x_0) + c_0 \le 0, \end{aligned}$$

where $\mathcal{N}_D = \{(x_0, x_1) \in \mathcal{N} \mid x_0 \in D_0, x_T = 0\}$. Recalling that $x_{-1} = 0$ by definition and that x_0 is deterministic since we have assumed $\mathcal{F}_0 = \{\Omega, \emptyset\}$, the problem can be written as

$$\begin{aligned} &\text{minimize} && \mathcal{V}_1(S_1(-x_0) + c_1) \quad \text{over} \quad x_0 \in D_0, \\ &\text{subject to} && S_0(x_0) + c_0 \le 0. \end{aligned}$$

In the linear case where $S_t(x, \omega) = s_t(\omega) \cdot x$, this becomes

$$\begin{aligned} &\text{minimize} && \mathcal{V}_1(c_1 - s_1 \cdot x_0) \quad \text{over} \quad x_0 \in D_0, \\ &\text{subject to} && s_0 \cdot x_0 \le -c_0, \end{aligned}$$

which is the single-period problem (ALM) *with initial wealth $w = -c_0$.*

With the traditional model of liquid markets, problem (2.1) can be written in terms of stochastic integrals.

Example 2.5 (Numeraire and stochastic integration) *Assume that there is a perfectly liquid asset (numeraire), say $0 \in J$, such that*

$$S_t(x,\omega) = s_t^0(\omega)x^0 + \tilde{S}_t(\tilde{x},\omega),$$
$$D_t(\omega) = \mathbb{R} \times \tilde{D}_t(\omega),$$

where $x = (x^0, \tilde{x})$, s^0 is a strictly positive adapted process and $\tilde{S}$ and $\tilde{D}$ are the cost process and the constraints for the remaining risky assets $\tilde{J} = J \setminus \{0\}$. Expressing all costs in terms of the numeraire, we may assume $s^0 \equiv 1$. We can then use the budget constraint to substitute out the numeraire from problem (2.1). Indeed, given an adapted trading strategy $\tilde{x}$ for the risky assets, the recursion

$$x_t^0 = x_{t-1}^0 - \tilde{S}_t(\Delta\tilde{x}_t) - c_t \quad t = 0,\dots,T-1,$$

with $x_{-1}^0 = 0$ defines an optimal strategy for trading the bank account. With this, the budget constraint holds as an equality for $t = 0,\dots,T-1$ and

$$x_{T-1}^0 = -\sum_{t=0}^{T-1} \tilde{S}_t(\Delta\tilde{x}_t) - \sum_{t=0}^{T-1} c_t.$$

Thus,

$$S_T(\Delta x_T) + c_T = \Delta x_T^0 + \tilde{S}_T(\Delta\tilde{x}_T) + c_T = x_T^0 + \sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) + \sum_{t=0}^{T} c_t.$$

Recalling that $x_T = 0$ for $x \in \mathcal{N}_D$, problem (2.1) can thus be written as

$$\text{minimize} \qquad \mathcal{V}_T\left(\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) + \sum_{t=0}^{T} c_t\right) \quad \text{over} \quad x \in \mathcal{N}_D,$$

Thus, in the presence of a numeraire, the timing of the payments is irrelevant; only their sum matters. Furthermore, in the linear case $\tilde{S}_t(\tilde{x},\omega) = \tilde{s}_t(\omega)\cdot\tilde{x}$, the cumulated trading costs can be written as the stochastic integral

$$\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) = \sum_{t=0}^{T} \tilde{s}_t \cdot \Delta\tilde{x}_t = -\sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1}.$$

We then recover a discrete-time version of the classical problem of optimal investment in perfectly liquid markets.

We will denote the optimal value of problem (ALM-d) by $\varphi(c)$. That is,

$$\varphi(c) := \inf_{x\in\mathcal{N}_D} \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t).$$

This is an extended real-valued function on the space $\mathcal{M}$ of adapted sequences of claims. In the totally risk averse case where $\mathcal{V}_t = \delta_{L^0_-}$ for all $t = 0, \dots, T$, we get $\varphi = \delta_{\mathcal{C}}$, where

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \ S_t(\Delta x_t) + c_t \leq 0 \ P\text{-a.s.} \quad t = 0, \dots, T\}, \tag{2.2}$$

the set of claim processes that can be *superhedged* without a cost. On the other hand, since the functions $\mathcal{V}_t$ are nondecreasing, we can always write the value function φ in terms of $\mathcal{C}$ as

$$\begin{aligned}\varphi(c) &= \inf_{x \in \mathcal{N}_D, d \in \mathcal{M}} \left\{ \sum_{t=0}^T \mathcal{V}_t(c_t + d_t) \,\middle|\, d_t = S_t(\Delta x_t) \right\} \\ &= \inf_{x \in \mathcal{N}_D, d \in \mathcal{M}} \left\{ \sum_{t=0}^T \mathcal{V}_t(c_t + d_t) \,\middle|\, d_t \geq S_t(\Delta x_t) \right\} \\ &= \inf_{d \in \mathcal{M}} \left\{ \sum_{t=0}^T \mathcal{V}_t(c_t + d_t) \,\middle|\, -d \in \mathcal{C} \right\},\end{aligned}$$

or more concisely as

$$\varphi(c) = \inf_{d \in \mathcal{C}} \mathcal{V}(c - d), \tag{2.3}$$

where $\mathcal{V}$ denotes the convex function on $\mathcal{M}$ defined by

$$\mathcal{V}(c) := \sum_{t=0}^T \mathcal{V}_t(c_t).$$

In (2.3), the variable d represents the part of liabilities that can be costlessly superhedged by trading in financial markets. If we ignore the financial market so that C coincides with the set $\mathcal{M}_-$ of nonpositive sequences of claims, we simply have $\varphi = \mathcal{V}$.

While the set $\mathcal{C}$ consists of the claims that can be superhedged without a cost, its *recession cone*

$$\mathcal{C}^\infty = \{c \in \mathcal{M} \,|\, \bar{c} + \alpha c \in \mathcal{C} \quad \forall \bar{c} \in \mathcal{C}, \ \forall \alpha > 0\}$$

consists of the claims that can be superhedged without a cost in unlimited amounts. Clearly, $\mathcal{M}_- \subseteq \mathcal{C}^\infty$ (one can always throw away as much cash as one wishes). The following lemma summarizes some key properties of the value function φ and the set $\mathcal{C}$.

Lemma 2.6 *The value function $\varphi : \mathcal{M} \to \overline{\mathbb{R}}$ is convex, $\varphi(0) \leq 0$ and*

$$\varphi(\bar{c} + c) \leq \varphi(\bar{c}) \quad \forall \bar{c} \in \mathcal{M}, \ c \in \mathcal{C}^\infty.$$

In particular, the set $\mathcal{C}$ is convex and $\mathcal{C}^\infty \subseteq \mathcal{C}$. If the trading costs S_t are sublinear and portfolio constraints D_t are conical, then $\mathcal{C}$ is a cone and $\mathcal{C}^\infty = \mathcal{C}$. In, in addition, $\mathcal{V}$ is sublinear, then φ is sublinear as well.

Proof. Convexity of $\mathcal{C}$ as well as the fact that $\mathcal{C}$ is a cone when $S_t(\cdot,\omega)$ are sublinear and $D_t(\omega)$ are conical is proved in Exercise 5 below. When $\mathcal{C}$ is a convex cone, we have $\mathcal{C}^\infty = \mathcal{C}$, by Exercise 1. The convexity of φ as well as the sublinearity in the last claim now follow from Exercise 6 once we write expression (2.3) as the infimal convolution of $\mathcal{V}$ and the indicator function $\delta_{\mathcal{C}}$ of the set $\mathcal{C}$. If $c \in \mathcal{C}^\infty$, then $\bar{c} + c \in \mathcal{C}$ whenever $\bar{c} \in \mathcal{C}$ so that

$$\inf_{d\in\mathcal{C}} \mathcal{V}(\bar{c} + c - d) = \inf_{d'+c\in\mathcal{C}} \mathcal{V}(\bar{c} - d') \leq \inf_{d'\in\mathcal{C}} \mathcal{V}(\bar{c} - d'),$$

which means that $\varphi(\bar{c} + c) \leq \varphi(\bar{c})$. □

Note that since $\mathcal{M}_- \subseteq \mathcal{C}^\infty$, the growth property in Lemma 2.6 implies that $\varphi(c^1) \leq \varphi(c^2)$ whenever $c^1 \leq c^2$ almost surely. This simply means that the agent is always better off with more money.

We will say that a claim $c \in \mathcal{M}$ is *redundant* if

$$c \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty),$$

i.e. if $\bar{c}+\alpha c \in \mathcal{C}$ for every $\bar{c} \in \mathcal{C}$ and every $\alpha \in \mathbb{R}$. In particular, arbitrary positive as well as negative multiples of redundant claims can be superhedged without a cost. By Lemma 2.6, the optimal value of (ALM-d) is invariant with respect to redundant claims. Redundancy generalizes the notion of "replicability" in the classical liquid market model as illustrated by Example 2.8 below. Redundant claims will play an important role in relating indifference swap rates to risk-neutral pricing in Section 2.4 below.

A market model (S, D) is said to satisfy the *no-arbitrage* condition if

$$\mathcal{C} \cap \mathcal{M}_+ = \{0\}. \tag{NA}$$

Violation of this condition would mean that there exist nonzero nonnegative claims that could be costlessly superhedged by trading in financial markets. The possibility of doing so is known as an *arbitrage opportunity.*

Example 2.7 *When $S_t(x) = s_t \cdot x$ and $D_t = \mathbb{R}^J$, a claim $c \in \mathcal{M}$ is redundant if there is an $x \in \mathcal{N}_D$ such that $s_t \cdot \Delta x_t + c_t = 0$. The converse holds under the no-arbitrage condition.*

Proof. By Lemma 2.6, $\mathcal{C}$ is now a cone and $\mathcal{C}^\infty = \mathcal{C}$. Thus, a claim $c \in \mathcal{M}$ is redundant iff $c \in \mathcal{C}$ and $-c \in \mathcal{C}$, which means that there exist $x^1, x^2 \in \mathcal{N}_D$ such that

$$s_t \cdot \Delta x_t^1 + c_t \leq 0 \quad \text{and} \quad s_t \cdot \Delta x_t^2 - c_t \leq 0 \quad \forall t. \tag{2.4}$$

If there is an $x \in \mathcal{N}_D$ such that $s_t \cdot \Delta x_t + c_t = 0$, then (2.4) holds with $x^1 = x$ and $x^2 = -x$. On the other hand, (2.4) implies $s_t \cdot \Delta(x_t^1 + x_t^2) \leq 0$, where equality must hold under (NA). Combining this equality with (2.4), we get

$$c_t \leq -s_t \cdot \Delta x_t^1 = s_t \cdot \Delta x_t^2 \leq c_t$$

so both x^1 and $-x^2$ satisfy the condition $s_t \cdot \Delta x_t + c_t = 0$. □

Example 2.8 (Numeraire and stochastic integration) *In the classical linear model with* $S_t(x) = x_0 + \tilde{s}_t \cdot \tilde{x}$ *and* $D_t = \mathbb{R}^J$ *(see Example 2.5), we have*

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} c_t \le \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1}\}$$

and a claim $c \in \mathcal{C}$ *is redundant if there exists an* $x \in \mathcal{N}_D$ *such that*

$$\sum_{t=0}^{T} c_t = \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1} \tag{2.5}$$

The converse holds under the (NA) *condition which, in this classical model, can be stated as*

$$x \in \mathcal{N}_D : \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1} \ge 0 \implies \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1} = 0.$$

Note that in the presence of a perfectly liquid cash-account, a claim is redundant in particular if $\sum_{t=0}^{T} c_t = 0$.

Proof. The expression for $\mathcal{C}$ follows from Example 2.5 by recalling that the value function φ becomes the indicator of $\mathcal{C}$ when $\mathcal{V}_t = \delta_{L^0_-}$ for every $t = 0, \dots, T$. By Lemma 2.6, $\mathcal{C}$ is now a cone and $\mathcal{C}^\infty = \mathcal{C}$. Thus, a claim $c \in \mathcal{M}$ is redundant iff $c \in \mathcal{C}$ and $-c \in \mathcal{C}$, which means that there exist $x^1, x^2 \in \mathcal{N}_D$ such that

$$\sum_{t=0}^{T} c_t \le \sum_{t=0}^{T-1} \tilde{x}^1_t \cdot \Delta\tilde{s}_{t+1} \quad \text{and} \quad -\sum_{t=0}^{T} c_t \le \sum_{t=0}^{T-1} \tilde{x}^2_t \cdot \Delta\tilde{s}_{t+1}. \tag{2.6}$$

If there is an $x \in \mathcal{N}_D$ such that (2.5) holds, then (2.6) hold with $x^1 = x$ and $x^2 = -x$. On the other hand, (2.6) implies

$$\sum_{t=0}^{T-1} (\tilde{x}^1_t + \tilde{x}^2_t) \cdot \Delta\tilde{s}_{t+1} \ge 0.$$

where equality must hold under (NA). Combining this equality with (2.6), we see that both $\tilde{x}^1$ and $-\tilde{x}^2$ satisfy (2.5). □

Exercises

A set K is said to be a *cone* if $\alpha x \in K$ for every $x \in K$ and $\alpha \ge 0$. Given a set $C \subseteq X$, the cone

$$C^\infty = \{y \,|\, x + \alpha y \in C \;\forall x \in C \;\forall \alpha > 0\}$$

is called the *recession cone* of C. The recession cone gives the directions in which C is “unbounded”.

1. Show that

   (a) the recession cone of a convex set is a convex cone.

   (b) if C is a convex cone and $x_1, x_2 \in C$, then $x_1 + x_2 \in C$.

   (c) if C is a convex cone, then $C^\infty = C$.

2. Show that if $\mathcal{R}$ is a convex risk measure in the sense of Example 1.4, then $\mathcal{A} = \operatorname{lev}_0 \mathcal{R}$ is a convex set with $L^\infty_- \subset \mathcal{A}^\infty$ (here $L^\infty_- = \{c \in L^\infty \,|\, c \leq 0 \text{ } P\text{-a.s.}\}$, the set of nonpositive essentially bounded random variables) and we have
$$\mathcal{R}(c) = \inf\{\alpha \in \mathbb{R} \,|\, c - \alpha \in \mathcal{A}\}.$$
Conversely, if $\mathcal{A} \subset L^\infty$ is any convex set with $L^\infty_- \subset \mathcal{A}^\infty$, then $\mathcal{R}(c) = \inf\{\alpha \in \mathbb{R} \,|\, c - \alpha \in \mathcal{A}\}$ is a convex risk measure.

3. Consider an optimal consumption problem where an agent is endowed with an adapted sequence $e \in \mathcal{M}$ of cash-flows. At each time $t = 0, \ldots, T$ the agent may consume part of the endowment and invest the remaining part in financial markets described by (S, D). Assume that the agent's preferences are described by a sequence of disutility functions $\mathcal{V}_t$ and show that this problem can be written as (ALM-d).

4. Assume that s is a componentwise strictly positive $(\mathcal{F}_t)_{t=0}^T$-adapted "market price" process as in Exercise 3 of Section 2.1. Write the asset-liability management problem (ALM-d) in terms of the mark-to-market values $h^j_t = s^j_t x^j_t$.

5. Show that $\mathcal{C}$ is a convex set containing the set $\mathcal{M}_-$ of nonpositive claim processes. Show also that if S is sublinear and D is conical, then $\mathcal{C}$ is a cone.

6. Let f_1 and f_2 be convex functions on a linear space X. Show that their *infimal convolution*
$$(f_1 \,\square\, f_2)(x) := \inf_{z \in X} \{f_1(x - z) + f_2(z)\}$$
is a convex function. Show also that if f_1 and f_2 are sublinear then $f_1 \,\square\, f_2$ is sublinear as well.

7. Show that the no-arbitrage condition can be expressed as follows: if $x \in \mathcal{N}_D$ is such that
$$S_t(\Delta x_t) \leq 0 \quad P\text{-a.s. } t = 0, \ldots, T$$
then
$$S_t(\Delta x_t) = 0 \quad P\text{-a.s. } t = 0, \ldots, T.$$

8. Let C be a convex set in a linear space and assume that C is *algebraically closed* in the sense that $\{\alpha \in \mathbb{R} \,|\, x + \alpha y \in C\}$ is a closed interval for every $x, y \in X$. Show that $y \in C^\infty$ if there exists even one $x \in C$ such that $x + \alpha y \in C$ for every $\alpha > 0$. In this case, we thus have the expression

$$C^\infty = \bigcap_{\alpha>0} \alpha(C - x)$$

which is independent of the choice of $x \in C$.

Further reading

Much of the theory of optimal investment has been developed in a continous-time setting, most famously by Merton [33] in a complete market model. Extensions to illiquid markets can be found e.g. in He and Pearson [19], Karatzas, Lehoczky, Shreve and Xu [27] and Kramkov and Schachermayer [29]. Illiquid extensions have been given in Hodges and Neuberger [21], Davis and Norman [10] and Çetin and Rogers [7]. A recent overview of optimal investment can be found in Rogers [45].

Besides the discrete-time and illiquidity effects, our model deviates from the above references in that it does not require the existence of a numeraire. This is significant from the practical point of view. Indeed, in practice, much of trading consists of exchanging sequences of cash-flows, exactly because one cannot transfer cash quite freely in time. Problem (ALM) can also be interpreted as an illiquid discrete-time version of the optimal consumption-investment problem with random endowment studied in Karatzas and Žitković [28]; see Exercise 3.

The notion of redundancy extends the notion of attainability from the classical linear market model to general nonlinear market models. Indeed, the condition in Example 2.8 means that $\sum_{t=0}^T c_t$ is "attainable at price zero" as defined e.g. in [12, Definition 2.2.1].

2.3 Accounting values

Analogously to the single-period model of Section 1.4, we define the accounting value for a financial liability as the least amount of initial capital that would allow an agent to pay the liability cash-flows at an acceptable risk. Liabilities are now described by sequences $c \in \mathcal{M}$ of cash flows and the least risk achievable by trading is given by the optimum value $\varphi(c)$ of (ALM-d). Defining $p^0 = (1, 0, \dots, 0)$,

$$\pi_s^0(c) = \inf\{\alpha \in \mathbb{R} \,|\, \varphi(c - \alpha p^0) \le 0\}$$

gives the least amount of initial capital one needs in order to construct a hedging strategy $x \in \mathcal{N}_D$ whose proceeds cover a sufficient part of the liabilities c so that the risk associated with the residual is no higher than the risk from doing nothing at all (recall that $\mathcal{V}_t(0) = 0$, by assumption). We will call $\pi_s^0(c)$ the *accounting value* of a liability $c \in \mathcal{M}$. The terms "reserve" and "reservation value" are also used, especially in the insurance industry.

If we ignore the financial market so that $\varphi = \mathcal{V}$, we simply have $\pi^0_s(c) = \inf\{\alpha \,|\, \mathcal{V}(c - \alpha p^0) \le 0\}$. If $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$ as in (2.1) and if the optimum value of (ALM-d) is attained, the accounting value for a claim $c \in \mathcal{M}$ is given by the optimum value of (exercise)

$$\begin{array}{lll} \text{minimize} & S_0(x_0) + c_0 \quad \text{over} \quad x \in \mathcal{N}_D, & \\ \text{subject to} & S_t(\Delta x_t) + c_t \le 0, \quad t = 1, \dots, T-1, & (2.7) \\ & \mathcal{V}_T(S_T(-x_{T-1}) + c_T) \le 0. & \end{array}$$

The corresponding solution $x \in \mathcal{N}_D$ gives an optimal *hedging strategy* for the liabilities $c \in \mathcal{M}$. The same idea is behind the classical Black–Scholes–Merton option pricing model where the price of an option is defined as the least amount of initial capital that allows for the construction of a trading strategy whose terminal value equals the payout of the option. Unlike in the Black–Scholes–Merton model, however, exact replication is usually impossible in practice so the accounting value of a liability depends one risk preferences concerning the unhedged part (the uncertain amount of claims that exceed the revenue from trading). Also, we have not assumed the existence of a numeraire so the natural domain of definition for π^0_s is the space $\mathcal{M}$ of sequences of cash-flows. This is important in practice where financial liabilities often have several payout dates. It is, however, often natural to assume that $c_0 = 0$ since a nonzero value of c_0 would just add directly to the required initial capital $\pi^0_s(c)$.

In the deterministic case, we recover the classical discounting principle from actuarial mathematics.

Example 2.9 (Actuarial best estimate) *Assume that c and S are deterministic with $J = \{0\}$ (only one traded asset) and $S_t(x, \omega) = x/P_t$ where P_t is the price at time 0 of a zero-coupon bond maturing at time t. If $\mathcal{V}_T(\alpha) > 0$ for positive constants α, problem (2.7) can be written as*

$$\begin{array}{ll} \text{minimize} & x_0 + c_0 \quad \text{over} \quad x \in \mathcal{N}_D, \\ \text{subject to} & \Delta x_t / P_t + c_t \le 0, \quad t = 1, \dots, T, \end{array}$$

where $\mathcal{N}_D$ now consists of sequences $(x_t)_{t=0}^T$ of real numbers with $x_T = 0$. This can be solved explicitly for $x_t = \sum_{s=t+1}^T P_s c_s$ and the optimum value

$$\pi^0_s(c) = \sum_{s=0}^T P_s c_s.$$

Such formulas are frequently encountered in actuarial mathematics where c_t is usually defined as the expected claims to be paid at time t. In that context, the above expression is known as the “best estimate” of the claims.

It should be noted, however, that the “best estimate” is appropriate only for the valuation of deterministic cash-flows. For uncertain ones, it can result in quite unreasonable values. For example, the “best estimate” of a European call option (obtained by defining c_T as the expected payout) is usually much higher *than its market or Black–Scholes value. Moreover, the "best estimate" has the unpleasant property that it increases when the interest rates decrease which is typical in adverse market conditions. This leads to the so called "procyclicality" problem in financial supervision. Nevertheless, the "best estimate" is still widely used in the insurance industry and it forms the basis of the regulatory standard in the Solvency II Directive 2009/138/EC of the European Parliament.*

While $\pi^0_s(c)$ gives the least amount of initial capital required for constructing an acceptable (in the sense of the preferences given by $\mathcal{V}$) hedging strategy for a liability $c \in \mathcal{M}$, the number

$$\pi^0_l(c) := \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\alpha p^0 - c) \leq 0\}$$

gives the greatest initial payment one could finance by trading in the financial markets if one were to receive $c \in \mathcal{M}$. If $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$ as in (2.1) and if the optimum value of (ALM-d) is attained, the accounting value for an asset $c \in \mathcal{M}$ is given by the optimum value of (exercise)

$$\begin{aligned} &\text{maximize} && c_0 - S_0(x_0) \quad \text{over} \quad x \in \mathcal{N}_D, \\ &\text{subject to} && S_t(\Delta x_t) - c_t \leq 0, \quad t = 1, \ldots, T-1, \\ & && \mathcal{V}_T(S_T(-x_{T-1}) - c_T) \leq 0. \end{aligned}$$

In the objective, c_0 is a payment one could cover immediately with c while $-S_0(x_0)$ is the amount of cash one can generating by "shorting" the traded assets. The constraints say that this short position is covered by future trading when receiving the remaining cash-flows of c. We will call $\pi^0_l(c)$ the *accounting value of an asset* $c \in \mathcal{M}$. Clearly, $\pi^0_l(c) = -\pi^0_s(-c)$.

Accounting values for assets are relevant e.g. in banks whose assets are divided into the *trading book* and the *banking book*. Assets in the banking book are typically illiquid assets such as individual loans without secondary markets. A bank usually holds such assets until maturity. Variable rates and credit risk make such a loan's cash flows c uncertain so the accounting value is not just the notional amount of the loan. Again, for liquidly traded assets, market prices should be used for valuation; see the last part of Theorem 2.10 below.

Just like in the single-period model, we can bound the accounting values between super- and subhedging costs. The *superhedging cost* of a $c \in \mathcal{M}$ is defined by

$$\pi^0_{\sup}(c) = \inf\{\alpha \in \mathbb{R} \,|\, c - \alpha p^0 \in \mathcal{C}\}.$$

It gives the least amount of initial capital required for delivering c without any risk of losing money. Analogously, we define the *subhedging cost* by

$$\pi^0_{\inf}(c) = \sup\{\alpha \in \mathbb{R} \,|\, \alpha p^0 - c \in \mathcal{C}\}.$$

Note that, while the accounting value $\pi^0_s(c)$ depends on the underlying probability measure P and the disutility functions $\mathcal{V}_t$, the super- and subhedging costs are essentially independent of such subjective factors since they depend on the probability measure P only through its null sets. Indeed, the accounting value is based on the optimum value function φ of the optimal investment problem (ALM-d) which depends on P and $\mathcal{V}_t$, while the superhedging cost $\pi^0_{\sup}$ is defined solely in terms of $\mathcal{C}$ which depends on P only up to its null sets. The following summarizes some basic properties of the accounting value.

Theorem 2.10 *The function π^0_s is convex, $\pi^0_s(0) \le 0$ and*

$$\pi^0_s(c + c') \le \pi^0_s(c) \quad \forall c \in \mathcal{M},\ \forall c' \in \mathcal{C}^\infty.$$

We always have $\pi^0_s(c) \le \pi^0_{\sup}(c)$. If $\pi^0_s(0) = 0$, then

$$\pi^0_{\inf}(c) \le \pi^0_l(c) \le \pi^0_s(c) \le \pi^0_{\sup}(c) \quad \forall c \in \mathcal{M}$$

with equalities throughout when $c - \bar{\alpha} p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for some $\bar{\alpha} \in \mathbb{R}$ and in this case, $\pi^0_s(c) = \bar{\alpha}$.

Proof. Defining $\mathcal{A} = \{c \in \mathcal{M} \,|\, \varphi(c) \le 0\}$, we have

$$\pi^0_s(c) = \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{A}\}.$$

Lemma 2.6 implies that $\mathcal{A}$ is a convex set with $\mathcal{C}^\infty$ in its recession cone. This implies the convexity of π_s; see Exercise 2 in Section 1.3. The monotonicity follows from the fact that if $c' \in \mathcal{C}^\infty$ and $c - \alpha p^0 \in \mathcal{A}$, then $c + c' - \alpha p^0 \in \mathcal{A}$ so

$$\pi^0_s(c + c') = \inf\{\alpha \,|\, c + c' - \alpha p^0 \in \mathcal{A}\} \le \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{A}\} = \pi^0_s(c).$$

Since $\mathcal{V}(0) = 0$, expression (2.3) implies $\varphi \le \delta_{\mathcal{C}}$ and thus,

$$\begin{aligned}
\pi^0_s(c) &= \inf\{\alpha \,|\, \varphi(c - \alpha p^0) \le 0\} \\
&\le \inf\{\alpha \,|\, \delta_{\mathcal{C}}(c - \alpha p^0) \le 0\} \\
&= \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{C}\} = \pi^0_{\sup}(c).
\end{aligned}$$

Since $\pi^0_{\inf}(c) = -\pi^0_{\sup}(-c)$, we must also have $\pi^0_{\inf}(c) \le -\pi^0_s(-c) = \pi^0_l(c)$. By convexity,

$$\pi^0_s(0) = \pi^0_s\Big(\frac{1}{2}c + \frac{1}{2}(-c)\Big) \le \frac{1}{2}\pi^0_s(c) + \frac{1}{2}\pi^0_s(-c)$$

so if $\pi^0_s(0) = 0$ we get $-\pi^0_s(-c) \le \pi^0_s(c)$, which completes the proof of the "sandwich" inequalities. If $c - \bar{\alpha} p^0 \in \mathcal{C} \cap (-\mathcal{C})$, we get $\pi^0_{\sup}(c) \le \bar{\alpha} \le \pi^0_{\inf}(c)$, which completes the proof. □

The convexity and growth properties of the accounting value π^0_s make it reminiscent of convex risk measures; see Example 1.4. Indeed, since $\mathcal{M}_- \subset \mathcal{C}^\infty$ we have $\pi^0_s(c^1) \le \pi^0_s(c^2)$ whenever $c^1 \le c^2$. However, the translation property 3 in Example 1.4 has no natural counterpart for claims with multiple payout dates in markets without a perfectly liquid numeraire asset; see Exercise 4 however.

The condition $\pi^0_s(0) = 0$ means that the accounting value for the zero claim is zero. This is quite a reasonable assumption form the regulator's point of view.

It holds, in particular, if $\mathcal{V}_0 = \delta_{L^0_-}$, the cost function S_0 is nondecreasing and if short selling is prohibited, i.e. if $D_0 \subseteq R^J_+$. In the totally risk averse case with $\mathcal{V}_t = \delta_{\mathbb{R}_-}$ for all t, the condition $\pi^0_s(0) = 0$ means that it is not possible to superhedge the claim αp^0 for $\alpha > 0$. In the classical linear model, such a condition is known as the "weak no-arbitrage" condition; see Exercise 4b in Section 1.4. As the name suggests, the weak no-arbitrage condition is a weaker condition than the no-arbitrage condition (NA); see Exercise 3 below.

In general, $\pi^0_s(c)$ depends on the probability measure P and the disutility functions $\mathcal{V}_t$ (views and preferences) but by the last part of Theorem 2.10, $\pi^0_s(c)$ is independent of such subjective factors when $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for some $\alpha \in \mathbb{R}$.

Example 2.11 (Numeraire and stochastic integration) *Consider again the classical model of unconstrained liquid markets in Example 2.8. We get*

$$\pi^0_{\sup}(c) = \inf\{\alpha \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} c_t \le \alpha + \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}.$$

Since now $\mathcal{C}^\infty = \mathcal{C}$, we have $c-\alpha p^0 \in \mathcal{C}\cap(-\mathcal{C})$ if and only if $c-\alpha p^0$ is redundant. By Example 2.8, this holds if there is an $x \in \mathcal{N}_D$ such that

$$\sum_{t=0}^{T} c_t = \alpha + \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}.$$

Under the no-arbitrage condition, the converse holds. We thus recover the replication-based pricing principle behind the Black–Scholes–Merton model. The accounting value π^0_s extends the replication argument to incomplete and inlliquid markets.

Proof. In the classical linear model, we have, by Example 2.8, that

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} c_t \le \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}$$

so

$$\begin{aligned}\pi^0_{\sup}(c) &= \inf\{\alpha \,|\, c - \alpha p^0 \in \mathcal{C}\}\\ &= \inf\{\alpha \,|\, \exists x \in \mathcal{N}_D : \sum_{t=0}^{T} (c_t - \alpha p^0_t) \le \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}.\end{aligned}$$

Since $p^0 = (1, 0, \ldots, 0)$, this gives the expression for $\pi^0_{\sup}(c)$. The rest follow from the characterization of redundancy in Example 2.8. □

Exercises

1. Show that when $T = 1$, $c_0 = 0$ and $S_t(x, \omega) = s_t(\omega) \cdot x$, problem (2.7) reduces to problem (1.1) in the one-period setting.

2. Show that if $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$ and if the optimum value of (ALM-d) is attained for every $c \in \mathcal{M}$, then $\pi^0_s(c)$ is given by the optimum value of (2.7) as claimed.

3. Show that the no-arbitrage condition (NA) implies the weak no-arbitrage condition. Give an example of a market model that satisfies the weak no-arbitrage condition but fails the no-arbitrage condition.

4. Consider an agent with $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$ and a market with a perfectly liquid numeraire asset as in Example 2.5.

   (a) Let $p \in \mathcal{M}$ be such that
   $$\sum_{t=0}^{T} p_t = 1 \quad P\text{-a.s.}$$
   Show that $\pi^0_s(c + \alpha p) = \pi^0_s(c) + \alpha$ for every $\alpha \in \mathbb{R}$.

   (b) Show that if $\mathcal{V}_T$ is a convex risk measure in the sense of Example 1.4, then $\pi^0_s = \varphi$.

Further reading

Our definition of the accounting value π^0_s corresponds to the usual definition of an "option price" as the least amount of initial cash needed for hedging the option. We have avoided the term "price" since that is more appropriate for the notion of indifference price to be studied in the next section; see also Section 1.5. Extensive treatments of superhedging can be found e.g. in [17] and [12]. More reasonable risk preferences are studied in a liquid multiperiod model in Föllmer and Schied [17, Chapter 8]. Illiquid generalizations can be found e.g. in Davis, Panas and Zariphopoulou [11]. The presented approach has been applied to the valuation of pension liabilities where in [20].

In the context of the classical linear market model, the condition that $c - \alpha p^0 \in \mathcal{C} \cap (-\mathcal{C})$ for some α generalizes the condition of "attainability"; see e.g. [17] and [12]. The notion of "weak no-arbitrage" condition has been studied in the context of perfectly liquid markets in [22] under the name "no-arbitrage of the second type" and in fixed-income markets with illiquidity effects in [13, 14].

2.4 Swap contracts

Much of trading in practice consists of exchanging sequences of cash-flows. This is the case in various swap and insurance contracts where both claims as well as premiums are paid over several points in time. The timing of payments matters since in the absence of a perfectly liquid bank account, one cannot transfer both positive as well as negative amounts of cash quite freely through time. This section extends the classical indifference pricing principle to general swap contracts in illiquid markets. We show that indifference swap rates lie between arbitrage bounds suitably generalized to account for nonlinear illiquidity effects and general premium processes.

Consider an agent whose current financial position is described by a sequence of liability payments $\bar{c} \in \mathcal{M}$ (recalling that negative liability payments are interpreted as income) and a possibility to enter a *swap contract* where the agent agrees to deliver a sequence $c \in \mathcal{M}$ of random payments in exchange for receiving another sequence proportional to a given *premium process* $p \in \mathcal{M}$. The lowest *swap rate* (premium rate) that would allow the agent to enter the contract without worsening his risk-return profile is given by

$$\pi_s(\bar{c}, p; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}.$$

Similarly,

$$\pi_l(\bar{c}, p; c) := \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + \alpha p - c) \leq \varphi(\bar{c})\}$$

gives the highest swap rate the agent would accept for taking the opposite side of the trade. Clearly, $\pi_l(\bar{c}, p; c) = -\pi_s(\bar{c}, p; -c)$. We will call $\pi_s(\bar{c}, p; c)$ and $\pi_l(\bar{c}, p; c)$ the *indifference swap rates*.

When $p = (1, 0, \dots, 0)$ and $c = (0, \dots, 0, c_T)$, we recover the traditional indifference price, i.e. the least initial payment that compensates for delivering a claim with single payment date. Even in the traditional setting, the value of $\pi_s(\bar{c}, p; c)$ still depends on the agent's existing liabilities $\bar{c}$ and it is, in general, different from the accounting value $\pi_s^0(c)$ studied in the previous section. In an *interest rate swap* (with equally spaced payment dates), one has $p = (N, \dots, N)$ and $c = (r_0 N, \dots, r_T N)$, where N is the notional and r_t the floating rate. In a *credit default swap*, the premium process p is a constant sequence until the default of the reference entity after which it is zero while the claim c is zero except at the first time t after default when it pays a fixed amount. *Collateralized debt obligations* also fit the above format. More examples can be found both in banking and the insurance industry.

The indifference swap rates can be bounded by the *super*- and *subhedging swap rates* defined by

$$\pi_{\sup}(p; c) = \inf\{\alpha \in \mathbb{R} \,|\, c - \alpha p \in \mathcal{C}^\infty\}$$

and

$$\pi_{\inf}(p; c) = \sup\{\alpha \in \mathbb{R} \,|\, \alpha p - c \in \mathcal{C}^\infty\},$$

where $\mathcal{C}^\infty$ is the recession cone of $\mathcal{C}$ defined in Section 2.2. Clearly, $\pi_{\inf}(p; c) = -\pi_{\sup}(p; -c)$. When $\mathcal{C}$ is a cone and $p = (1, 0, \dots, 0)$, we have $\mathcal{C}^\infty = \mathcal{C}$ and the functions $\pi_{\sup}(p; \cdot)$ and $\pi_{\inf}(p; \cdot)$ coincide with the super- and subhedging costs $\pi^0_{\sup}$ and $\pi^0_{\inf}$ considered in the previous section.

Note that, while the indifference swap rates depend on the underlying probability measure P (the agent's views concerning the uncertain future), the disutility functions $\mathcal{V}_t$ (the agent's risk preferences) and the liabilities $\bar{c}$ (the agent's financial position), the super- and subhedging swap rates are essentially independent of such subjective factors since they depend on the underlying probability measure P only through its null sets.

The following summarises the main properties of indifference swap rates.

Theorem 2.12 *Let $\bar{c}, p \in \mathcal{M}$. The function $\pi_s(\bar{c}, p; \cdot)$ is convex, $\pi_s(\bar{c}, p; 0) \leq 0$ and*

$$\pi_s(\bar{c}, p; c + c') \leq \pi_s(\bar{c}, p; c) \quad \forall c \in \mathcal{M},\ \forall c' \in \mathcal{C}^\infty.$$

We always have $\pi_s(\bar{c}, p; c) \leq \pi_{\sup}(p; c)$ and if $\pi_s(\bar{c}, p; 0) = 0$, then

$$\pi_{\inf}(p; c) \leq \pi_l(\bar{c}, p; c) \leq \pi_s(\bar{c}, p; c) \leq \pi_{\sup}(p; c) \quad \forall c \in \mathcal{M}$$

with equalities throughout when $c - \bar{\alpha}p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\bar{\alpha} \in \mathbb{R}$ and in this case, $\pi_s(\bar{c}, p; c) = \bar{\alpha}$.

Proof. Defining $\mathcal{A}(\bar{c}) = \{c \in \mathcal{M} \mid \varphi(\bar{c} + c) \leq \varphi(\bar{c})\}$, we have

$$\pi_s(\bar{c}, p; c) = \inf\{\alpha \mid c - \alpha p \in \mathcal{A}(\bar{c})\}.$$

Lemma 2.6 implies that $\mathcal{A}(\bar{c})$ is a convex set with $\mathcal{C}^\infty$ in its recession cone. The convexity of $\pi_s(\bar{c}, p; \cdot)$ thus follows from Exercise 2 in Section 1.3. If $c - \alpha p \in \mathcal{A}(\bar{c})$ and $c' \in \mathcal{C}^\infty$, we have $c + c' - \alpha p \in \mathcal{A}(\bar{c})$, so

$$\pi_s(\bar{c}, p; c + c') = \inf\{\alpha \mid c + c' - \alpha p \in \mathcal{A}(\bar{c})\} \leq \inf\{\alpha \mid c - \alpha p \in \mathcal{A}(\bar{c})\} = \pi_s(\bar{c}, p; c).$$

By convexity,

$$\pi_s(\bar{c}, p; 0) \leq \frac{1}{2}\pi_s(\bar{c}, p; c) + \frac{1}{2}\pi_s(\bar{c}, p; -c),$$

so that $\pi_l(\bar{c}, p; c) \leq \pi_s(\bar{c}, p; c)$ when $\pi_s(\bar{c}, p; 0) = 0$. If $c - \alpha p \in \mathcal{C}^\infty$, Lemma 2.6 gives $\varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})$ and thus

$$\pi_s(\bar{c}, p; c) = \inf\{\alpha \mid \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\} \leq \inf\{\alpha \mid c - \alpha p \in \mathcal{C}^\infty\} = \pi_{\sup}(p; c).$$

Since $\pi_{\inf}(p; c) = -\pi_{\sup}(p; -c)$ and $\pi_l(\bar{c}, p; c) = -\pi_s(\bar{c}, p; -c)$ we also get $\pi_{\inf}(p; c) \leq \pi_l(\bar{c}, p; c)$, which completes the proof of the inequalities. If $c - \bar{\alpha}p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$, we get $\pi_{\sup}(p; c) \leq \bar{\alpha}$ and $\pi_{\inf}(p; c) \geq \bar{\alpha}$ which completes the proof. □

The condition $\pi_s(\bar{c}, p; 0) = 0$ means that $\varphi(\bar{c} - \alpha p) > \varphi(\bar{c})$ for all $\alpha < 0$, or in other words, that one cannot deliver strictly positive multiples of p without worsening the optimal value of (ALM-d). The inequality $\pi_l(\bar{c}, p; c) \leq \pi_s(\bar{c}, p; c)$ means that two agents with identical characteristics have no incentive to enter the swap contract with each other. Differences in the financial position, views and/or risk preferences (as described by $\bar{c}$, P and $\mathcal{V}_t$, respectively) may provide incentive for trading. For example, two agents with opposing views on the future development of a foreign currency may be willing to enter a currency futures contract with each other. Even if they have similar views, they may be willing to enter the contract if they have different exposures to the currency risk (different $\bar{c}$). Recall also the example on wheat futures from the beginning of Section 1.5.

In the terminology of Section 2.2, the condition $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ in the last part of Theorem 2.12 means that the sequence $c - \bar{\alpha}p$ is *redundant*. The theorem implies that if $c - \alpha p$ is redundant for some α, then the indifference swap rates equal $\pi_{\sup}(p; c)$ which is independent of the subjective characteristics of the agent. Moreover, combining Theorems 2.12 and 2.10 gives the following.

Corollary 2.13 *When $\mathcal{C}$ is a cone, $p = (1, 0, \dots, 0)$, $\pi_s^0(0) = 0$ and $\pi_s(\bar{c}, p; 0) = 0$, the indifference swap rates coincide with the accounting values if $c - \alpha p$ is redundant for some α.*

Proof. When $\mathcal{C}$ is a cone and $p = (1, 0, \dots, 0)$, we have $\pi_{\sup}(p; \cdot) = \pi^0_{\sup}$ and $\pi_{\inf}(p; \cdot) = \pi^0_{\inf}$), so the claim follows from Theorems 2.12 and 2.10. □

The convexity of $\pi_{\sup}(p; \cdot)$ and the concavity of $\pi_{\inf}(p; \cdot)$ imply that swap rates are linear in c on the linear subspace of claims c such that $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$; see Exercise 1 below. The linearity follows from the fact that a function which is both concave and convex and vanishes at the origin must be linear. In general, however, both swap rates and accounting values are nonlinear functions of $c \in \mathcal{M}$. The convexity of $\pi_s(\bar{c}, p; \cdot)$ corresponds to the classical *diversification principle*: the price of a convex combination of two claims should be no higher than the convex combination of the prices. Also, the indifference swap rate for two units of a claim may be strictly higher than twice the indifference swap rate for one unit.

A market model is *complete* if $\mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ is a *maximal linear subspace* of $\mathcal{M}$, i.e. if for any $c \in \mathcal{M}$ and $p \notin \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ there is an $\alpha \in \mathbb{R}$ such that $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$. In a complete market model, the indifference swap rate always coincides with the superhedging swap rate provided $\pi_s(\bar{c}, p; 0) = 0$. Indeed, by Lemma 2.6, $\pi_s(\bar{c}, p; 0) = 0$ implies $p \notin \mathcal{C}^\infty$ (why?) so in a complete market model, the last condition of Theorem 2.12 is satisfied for all $c \in \mathcal{M}$. Completeness, however, is satisfied only by some special instances of perfectly liquid market models such as the binomial model with one risky asset (or, more generally, d-nomial model with $d-1$ risky assets; see [17, Chapter 5.4]) or the continuous-time Black–Scholes model. Realistic market models are typically far from being complete.

Remark 2.14 *If there is an arbitrage-free price process s such that $s_t \cdot x < S_t(x)$ for every $x \neq 0$ and t, then $\mathcal{C}^\infty \cap (-\mathcal{C}^\infty) = \{0\}$. Indeed, if $c \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ we also have $c \in \mathcal{C} \cap (-\mathcal{C})$, which means that there exist $x^1, x^2 \in \mathcal{N}_D$ such that $S_t(\Delta x_t^1) + c_t \leq 0$ and $S_t(\Delta x_t^2) - c_t \leq 0$ and thus,*

$$s_t \cdot \Delta(x_t^1 + x_t^2) \leq S_t(\Delta x_t^1) + S_t(\Delta x_t^2) \leq 0,$$

with the first inequality being strict unless $\Delta x_t^1 = \Delta x_t^2 = 0$. If s is arbitrage-free, we must have $\Delta x_t^1 = \Delta x_t^2 = 0$ (see Exercise 7 in Section 2.2) and thus, since $S_t(0) = 0$, we must have $c = 0$.

Exercises

1. Show that $\mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ is a linear space.

2. Show that if $p \in \mathcal{M}$ is such that $\varphi(c+p) > \varphi(c)$ for some $c \in \mathcal{M}$, then $p \notin \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$.

3. Consider a market with a single asset which is perfectly liquid and whose unit price is constant 1. Assume that the premium sequence in a swap contract satisfies $E\sum_t p_t > 0$. Show that for an agent with $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$ and $\mathcal{V}_T(c) = Ec$, the indifference swap rate can be expressed as

$$\pi_s(\bar{c}, p; c) = \frac{E\sum_{t=0}^T c_t}{E\sum_{t=0}^T p_t}.$$

   This is the classical "risk neutral" approximation of a swap rate.

4. Consider a forward rate agreement with 3-month LIBOR as the underlying rate, effective date $t_0 = 3$ and termination date $t_1 = 6$ months. Formulate the indifference forward rates for (a) a fixed rate payer (b) fixed rate receiver.

5. Consider an agent with $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$ and a market with a perfectly liquid numeraire asset as in Example 2.5. Show that

   (a) if $c' \in \mathcal{M}$ is such that

$$\sum_{t=0}^T c'_t = \sum_{t=0}^T p_t \quad P\text{-a.s.}$$

   then the indifference swap rate has the translation property

$$\pi_s(\bar{c}, p; c + \alpha c') = \pi_s(\bar{c}, p; c) + \alpha \quad \forall \alpha \in \mathbb{R}.$$

   (b) If $\mathcal{V}_T$ is a risk measure in the sense of Example 1.4 and $\sum_{t=0}^T p_t$ is deterministic, then

$$\pi_s(\bar{c}, p; c) = \frac{\varphi(\bar{c}+c) - \varphi(\bar{c})}{\sum_{t=0}^T p_t}$$

   so one can avoid the line search in the definition of π_s.

Further reading

Indifference pricing has been studied extensively in incomplete financial markets; see the references at the end of Section 1.5. It has mostly, however, been restricted to models with a numeraire and to claims with a single payout date. The more general framework described here was introduced in [38]. Superhedging swap rates were first studied in [36]. An extensive treatment of recession cones in finite-dimensional spaces can be found in Sections 8 and 9 of [40].

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